Prove by contradiction that there are no rational roots to the equation $x^3 + x + 1 = 0$

Assume the opposite…

Assume that the roots, x are rational

$$x = \frac{\mathrm{b}}{\mathrm{a}}, \mathrm{a}, \mathrm{b} \in \mathbb{Z}, \mathrm{b} \neq 0$$

Since, x is in its simplest form, a and b are coprime

$$\left(\frac{\mathrm{b}}{\mathrm{a}}\right)^3 + \left(\frac{\mathrm{b}}{\mathrm{a}}\right) + 1 = 0$$

$$\frac{\mathrm{b}^3}{\mathrm{a}^3} + \frac{\mathrm{b}}{\mathrm{a}} + 1 = 0$$

$$\mathrm{b}^3 + \mathrm{a}^2\mathrm{b} + \mathrm{a}^3 = 0$$

Let's consider all 4 possible cases for a and b

1) a and b are even
2) a and b are odd
3) a is odd and b is even
4) a is even and b is odd

1) a and b are even — This is not possible, since a and b are coprime

2) a and b are odd — $\mathrm{b}^3 + \mathrm{a}^2 b + \mathrm{a}^3 = odd + odd + odd = odd$

This is not possible, since 0 is not odd

3) a is odd and b is even — $\mathrm{b}^3 + \mathrm{a}^2 b + \mathrm{a}^3 = even + even + odd = odd$

This is not possible, since 0 is not odd

5) a is even and b is odd — $\mathrm{b}^3 + \mathrm{a}^2 b + \mathrm{a}^3 = odd + odd + even = odd$

This is not possible, since 0 is not odd

Hence, there is a contradiction

We have proved that there are no rational roots to the equation $x^3 + x + 1 = 0$