Prove that $\sqrt{3}$ is irrational

Let $r = \sqrt{3}$

Assume that the opposite is true

Assume that r is rational

$$r = \frac{a}{b}, a, b \in \mathbb{Z}, b \neq 0$$

r is a fraction in its lowest terms

$$r^2 = \frac{a^2}{b^2} = 3$$

$$a^2 = 3b^2$$

Consider that b^2 is even,
then $3b^2$ is even,
so $a^2$is even

Therefore, a and b must be even

If a and b are both even, then $\frac{a}{b}$ can be simplified by dividing through by a common factor of 2.

This is a contradiction

We need to consider two cases

1) where b^2 is odd
2) where b^2 is even

Consider that b^2 is odd,
then $3b^2$ is odd,
so $a^2$is odd

Therefore, a and b must be odd

Let $a = 2m - 1$
Let $b = 2n - 1$

Therefore

$$a^2 = 3b^2$$

$$(2m - 1)^2 = 3(2n - 1)^2$$

$$4m^2 - 4m + 1 = 3(4n^2 - 4n + 1)$$

$$4m^2 - 4m + 1 = 12n^2 - 12n + 3$$

$$4m^2 - 4m = 12n^2 - 12n + 2$$

$$2m^2 - 2m = 6n^2 - 6n + 1$$

$$2(m^2 - m) = 2(3n^2 - 3n) + 1$$

Notice that the LHS is even and the RHS is odd

This is a contradiction. $\sqrt{3}$ is not rational. We have proved that $\sqrt{3}$ is irrational.