Binomial Theorem

The Binomial Theorem is used for expanding brackets in the form$(a+b)^n$

$$(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \cdots + \binom{n}{n}a^0 b^n$$

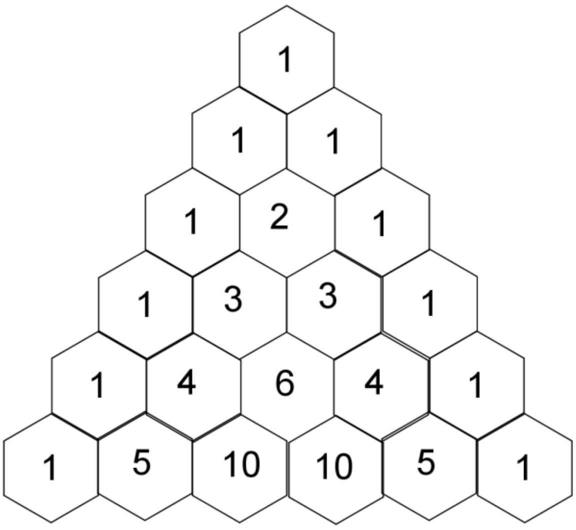
Example

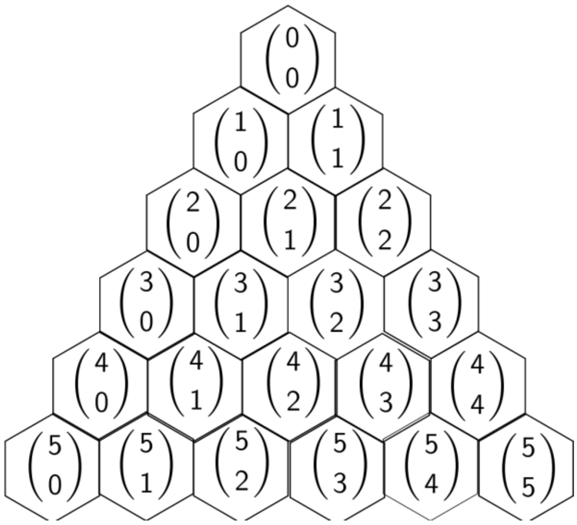
You can use either Pascal's Triangle or combinations to find the coefficients

$$(2x - \frac{3}{x})^4 = 1(2x)^4 + 4(2x)^3\left(-\frac{3}{x}\right)^1 + 6(2x)^2\left(-\frac{3}{x}\right)^2 + 4(2x)^1\left(-\frac{3}{x}\right)^3 + 1\left(-\frac{3}{x}\right)^4$$

$$(2x - \frac{3}{x})^4 = (2x)^4 + \binom{4}{1}(2x)^3\left(-\frac{3}{x}\right)^1 + \binom{4}{2}(2x)^2\left(-\frac{3}{x}\right)^2 + \binom{4}{3}(2x)^1\left(-\frac{3}{x}\right)^3 + \left(-\frac{3}{x}\right)^4$$

We can see that the numbers in rows of Pascal's triangle are combinations:





Extending the Binomial Theorem to Fractional and Negative Indices

We can use the Binomial Theorem to expand $(a+b)^n$ when n is not just a positive integer.

This form of the formula is useful:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

We need to write the expansion in the correct form:

$$(a+b)^n = \left(a\left(1+\frac{b}{a}\right)\right)^n = a^n\left(1+\frac{b}{a}\right)^n$$

Example

$$(2+x)^{-1} = \left(2\left(1+\frac{x}{2}\right)\right)^{-1} = 2^{-1}\left(1+\frac{x}{2}\right)^{-1}$$

$$= \frac{1}{2}\left(1 + (-1)\left(\frac{x}{2}\right) + \frac{(-1)(-2)}{2!}\left(\frac{x}{2}\right)^2 + \frac{(-1)(-2)(-3)}{3!}\left(\frac{x}{2}\right)^3 + \cdots\right)$$

$$= \frac{1}{2}\left(1 - \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8} + \cdots\right)$$

$$= \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16} + \cdots$$

This infinite series is only valid when $\left|\frac{x}{2}\right| < 1$ or $|x|<2$