5.8 Advanced Differentiation

Question Paper

Course	DP IB Maths
Section	5. Calculus
Topic	5.8 Advanced Differentiation
Difficulty	Very Hard

Time allowed: 130

Score: /105

Percentage: /100

Question 1

Let $f(x) = (ax + b)^2$, where $a, b \in \mathbb{R}$ are constants with $a \neq 0$.

By differentiating from first principles, show that $f'(x) = 2a(ax + b)$.

[5 marks]

Question 2a

Consider the function f defined by

$$f(x) = e^{-\frac{1}{2}x} \sin\left(\frac{\sqrt{3}}{2}x\right)$$

(a)
By first calculating $f'(x)$ and $f''(x)$, show that $f'''(x) = f(x)$.

[6 marks]

Question 2b

(b)
Write down the value of $f^{(29)}(0)$.

[2 marks]

Question 3a

(a)
Find the derivative of the function $f(x) = \tan(\ln x) + \arctan(e^x)$.

[4 marks]

Question 3b

(b)
Given that $g(x) = 27^{x+1} + \frac{1}{3}\log_3 x^6$, find $g'(x)$. Simplify your answer as far as possible.

[4 marks]

Question 3c

(c)
Let h be the function defined by $h(x) = \arcsin(\cos x)$. Show that

$$h'(x) = \begin{cases} 1, & (2k-1)\pi < x < 2k\pi \\ -1, & 2k\pi < x < (2k+1)\pi \\ \text{undefined}, & x = k\pi \end{cases}$$

where $k \in \mathbb{Z}$.

[5 marks]

Question 4

Use differentiation to show that $y = 1 + \tan x^3$ is a solution to the equation

$$\frac{d^2y}{dx^2} = 18x^4(y^3 - 3y^2 + 4y - 2) + 6x(y^2 - 2y + 2)$$

[7 marks]

Question 5a

Consider the curve defined by $y = \arccos(\ln x)$, for values of x satisfying $\frac{1}{e} \leq x \leq e$.

(a)
Show that

$$y'' = \frac{1}{x^2\sqrt{1-\ln^2 x}}\left(\frac{1-\ln x-\ln^2 x}{1-\ln^2 x}\right)$$

[5 marks]

Question 5b

(b)

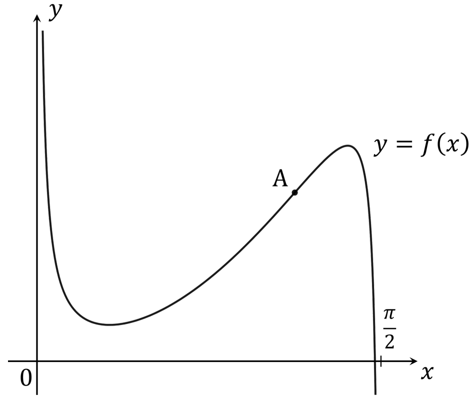
Given that the curve has exactly one point of inflection, show that that point of inflection occurs when $x = e^{\frac{1}{\phi}}$, where $\phi = \frac{1+\sqrt{5}}{2}$ is the so-called 'golden ratio'.

[7 marks]

Question 6

Consider the function f defined by $f(x) = 8x^2 + \cot 2x,\ 0 < x < \frac{\pi}{2}$.

The following diagram shows the graph of the curve $y = f(x)$:



The point marked A is the inflection point of the graph.

Determine the exact coordinates of the point where the normal to the graph at point A intersects the y-axis.

[11 marks]

Question 7a

For each of the following, find $\frac{\mathrm{d}y}{\mathrm{d}x}$ by differentiating implicitly with respect to x.

(a)
$$\frac{x^2}{3\sqrt{y}} + \frac{y^2}{4\sqrt{x}} = 1$$

[3 marks]

Question 7b

(b)
$$\sin(xy) = (x - 2y)^5$$

[3 marks]

Question 7c

(c)
$\sqrt{e^{-2x} - 2e^{-x-y} + e^{-2y}} = \pi$

[3 marks]

Question 8a

A curve is described by the equation

$$\frac{x - 2y}{(xy)^2} = k$$

where $k \in \mathbb{R}$ is a constant.

(a)
Use implicit differentiation to show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{xy - 4y^2}{2xy - 2x^2}$$

[5 marks]

Question 8b

For a particular value of k, the curve goes through the point $(-1, -1)$.

(b)
Find the value of k.

[2 marks]

Question 8c

(c)
Find the equation of the
(i)
tangent
(ii)
normal
to the curve at the point $(2, -1)$.

[4 marks]

Question 9a

Two observers, Pamela and Quinlan, are standing at points P and Q respectively watching a hot air balloon take off. The balloon takes off from point O, which is in between points P and Q and is such that points P, O and Q all lie on a straight horizontal line.

Let be the distance OP, and let D_p be the distance between point P and the balloon at any time t. Similarly let q be the distance OQ, and let D_q be the distance between point Q and the balloon at any time t. Let h be the height of the balloon above the ground at any time t. The balloon ascends vertically upwards, but its velocity during the ascent is not necessarily constant. All distances are measured in metres, and all times in seconds.

(a)

Show that an expression for $\frac{\mathrm{d}D_p}{\mathrm{d}t}$ can be written solely in terms of p, q and D_q.

[6 marks]

Question 9b

Quinlan is standing a distance of 50 metres from where the balloon takes off. At a certain moment in time, the balloon is at a distance of 112 metres from point Q and the distance between the balloon and point Q is increasing at a rate of 1.79 $\mathrm{m\,s^{-1}}$. At the same moment in time the distance between point P and the balloon is increasing at a rate of 1.05 $\mathrm{m\,s^{-1}}$.

(b)
Use the above information and the results of part (a) to determine the distance that Pamela is standing from the point where the balloon takes off.

[2 marks]

Question 9c

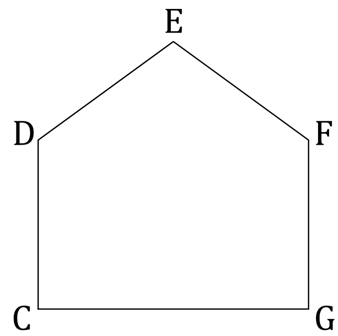
A third observer, Rhydderch, is standing at point R. Point R is on the same side of point O as point P is, and it lies on the same horizontal line as points O, P and Q. At the same moment described above, the distance between the balloon and point R is increasing at a rate of less than 0.8 metres per second.

(c)
Find an inequality to express the minimum distance PR between the point where Rhydderch is standing and the point where Pamela is standing.

[3 marks]

Question 10a

In the diagram below, CDEFG is a pentagon made up of a rectangle CDFG, to one side of which an isosceles triangle DEF has been appended. In addition sides CD and FG of the rectangle are the same length as the equal sides DE and EF of the triangle.



The pentagon is intended to represent the cross-section of a new building, and the architect would like the area of the pentagon, A, to be the maximum possible for any given perimeter, P.

Let $\mathrm{CG} = 2x$ units and let $\mathrm{DE} = y$ units.

(a)

By first finding the derivative $\frac{\mathrm{d}P}{\mathrm{d}x}$ in terms of x and y, work out the value of the derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$.

[4 marks]

Question 10b

(b)

By considering the derivative $\frac{\mathrm{d}A}{\mathrm{d}x}$, show that when the area is maximal for a given perimeter the following equation must hold:

$$20x^4 - 8x^3y - 3x^2y^2 + 12xy^3 - 12y^4 = 0$$

[8 marks]

Question 10c

(c)
Hence determine (i) the ratio of x to y (in the form $k:1$ for some k to be determined) that gives the maximum area for a given perimeter, and (ii) the maximum possible area for a pentagon of the above form with a perimeter of 100 metres.

[6 marks]