5.10 Differential Equations

Question Paper

Course	DP IB Maths
Section	5. Calculus
Topic	5.10 Differential Equations
Difficulty	Medium

Time allowed: 110

Score: /87

Percentage: /100

Question 1

Consider the first-order differential equation

$$\frac{dy}{dx} - 5x^4 = 3$$

Solve the equation given that $y = 40$ when $x = 2$, giving your answer in the form $y = f(x)$.

[5 marks]

Question 2a

Use separation of variables to solve each of the following differential equations for y:

a)

$$\frac{dy}{dx} = \frac{4x^2}{y^4}$$

[4 marks]

Question 2b

b)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (x^2+1)e^{-y}$$

[1 mark]

Question 3a

Use separation of variables to solve each of the following differential equations for which satisfies the given boundary condition:

a)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = xy^2; \quad y(2) = 1$$

[1 mark]

Question 3b

b)

$$(x+3)\frac{\mathrm{d}y}{\mathrm{d}x} = \sec y; \quad y(-2) = \frac{3\pi}{2}$$

[5 marks]

Question 4a

At any point in time, the rate of growth of a colony of bacteria is proportional to the current population size. At time $t = 0$ hours, the population size is 5000.

a)
Write a differential equation to model the size of the population of bacteria.

[1 mark]

Question 4b

After 1 hour, the population has grown to 7000.

b)
By first solving the differential equation from part (a), determine the constant of proportionality.

[6 marks]

Question 4c

c)
(i)
Show that, according to the model, it will take exactly $\frac{\ln 20}{\ln 7 - \ln 5}$ hours (from $t = 0$) for the population of bacteria to grow to $100\ 000$.

(ii)
Confirm your answer to part (c)(i) graphically.

[5 marks]

Question 5a

After clearing a large forest of malign influences, a wizard introduces a population of 100 unicorns to the forest. According to the wizard's mathemagicians, the population of unicorns in the forest may be modelled by the logistic equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = 0.0006\,P(250 - P)$$

where t is the time in years after the unicorns were introduced to the forest.

a)
Show that the population of unicorns at time t years is given by

$$P(t) = \frac{500\mathrm{e}^{0.15t}}{3 + 2\mathrm{e}^{0.15t}}$$

[8 marks]

Question 5b

b)
Find the length of time predicted by the model for the population of unicorns to double in size.

[3 marks]

Question 5c

c)
Determine the maximum size that the model predicts the population of unicorns can grow to.

[2 marks]

Question 6a

a)
Show that

$$x^2\frac{\mathrm{d}y}{\mathrm{d}x} = xy + 2x^2$$

is a homogeneous differential equation.

[2 marks]

Question 6b

b)
Using the substitution $v = \frac{y}{x}$, show that the solution to the differential equation in part (a) is

$$y = 2x\ln|x| + cx$$

where c is a constant of integration.

[4 marks]

Question 7a

a)

Use the substitution $v = \frac{y}{x}$ to show that the differential equation

$$y' = \frac{y^2}{x^2} - \frac{y}{x} + 1$$

may be rewritten in the form

$$v' = \frac{(v-1)^2}{x}$$

[3 marks]

Question 7b

b)

Hence use separation of variables to solve the differential equation in part (a) for which satisfies the boundary condition $y(1) = \frac{2}{3}$. Give your answer in the form $y = f(x)$.

[5 marks]

Question 8a

Consider the differential equation

$$y' + 2xy = (4x + 2)e^{x}$$

a)
Explain why it would be appropriate to use an integrating factor in attempting to solve the differential equation.

[2 marks]

Question 8b

b)
Show that the integrating factor for this differential equation is e^{x^2}.

[2 marks]

Question 8c

c)
Hence solve the differential equation.

[5 marks]

Question 9

Use an integrating factor to solve the differential equation

$$(x+3)\frac{\mathrm{d}y}{\mathrm{d}x} - 4y = (x+3)^6$$

for y which satisfies the boundary condition $y(-2) = 0$.

[7 marks]

Question 10a

Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x} + 1$$

with the boundary condition $y(1) = 0$.

a)
Apply Euler's method with a step size of $h = 0.2$ to approximate the solution to the differential equation at $x = 2$.

[3 marks]

Question 10b

b)

(i)

Explain what method you could use to solve the above differential equation analytically (i.e., exactly).

(ii)

The exact solution to the differential equation with the given boundary condition is $y = x \ln x$. Compare your approximation from part (a) to the exact value of the solution at $x = 2$.

[4 marks]

Question 10c

c)

Explain how the accuracy of the approximation in part (a) could be improved.

[1 mark]

Question 11a

A particle moves in a straight line, such that its displacement x at time t is described by the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{te^{3t^2}+1}{4x^2}, \quad t \geq 0$$

At time $t = 0$, $x = \frac{1}{2}$.

(a) By using Euler's method with a step length of 0.1, find an approximate value for x at time $t = 0.3$.

[3 marks]

Question 11b

(b)
(i)
Solve the differential equation with the given boundary condition to show that

$$x = \frac{1}{2}\sqrt[s]{e^{3t^2}+6t}$$

(ii)
Hence find the percentage error in your approximation for x at time $t = 0.3$.

[5 marks]