2.6 Further Modelling with Functions

Question Paper

Course	DP IB Maths
Section	2. Functions
Topic	2.6 Further Modelling with Functions
Difficulty	Medium

Time allowed: 130

Score: /104

Percentage: /100

Question 1a

The fare of a taxi ride starts at \$2.50 and increases by \$4.15 per km, for the first 5.5 km. After 5.5 km the rate charged per km decreases by \$0.50.

(a) Write down a piecewise function that models the fare of the taxi ride and show that the function is continuous when the taxi ride reaches 5.5 km.

[4 marks]

Question 1b

(b) Calculate the distance of a taxi ride that costs \$15.

[2 marks]

Question 1c

On a particular ride, a taxi driver accidentally takes a wrong turn 2.8 km into the ride that extends the journey by 2.1 km. Since the driver cannot switch off the fare calculator in the taxi without losing the details of the ride he decides to calculate the fair price for his customer, by removing the cost from the extra distance travelled. The total length of the journey, including the 2.1 km detour, was 9.4 km.

(c) Calculate the proper fare the customer should pay, having appropriately discounted the distance of the detour.

[3 marks]

Question 2a

The average temperature, T °C, of a city in winter every day can be modelled by the function

$$T(t) = a\sin(bt + k\pi) + d\,, \qquad 0 \le t < 24,$$

where t is the time, in hours after midnight, a and d are constants, and b is measured in radians.

(a) Find the value of b. Give your answer in terms of π.

[2 marks]

Question 2b

The average daily temperature range in the city is 6°C. The average maximum daily temperature is 2 °C and the average minimum daily temperature is at 3:00 am.

(b) Find the values of a, k and d.

[3 marks]

Question 2c

(c) Sketch the graph of T and label any intersections with the axes, local maximums and local minimums.

[4 marks]

Question 3a

The daily distance covered by an animal depends on the weather, and so varies each month. The following model represents the average daily distance, D km, covered by the animal each month.

$$D(t) = a\cos(bt + k) + d\,, \qquad 0 \leq t < 12,$$

Where t is the time, in months after the beginning of the year, a and d are constants, and b is measured in degrees.

(a) Find the value of b.

[2 marks]

Question 3b

The range of D is 22 km, the maximum daily distance covered is 29 km and the minimum daily distance covered occurs when $t = 6.5$.

(b) Find the values of a, k and d.

[3 marks]

Question 3c

(c) Sketch the graph of D and label any intersections with the axes, local maximums and local minimums.

[4 marks]

Question 3d

(d) Given that the animal covers the most distance in summer, state which hemisphere the animal is likely to live in.

[1 mark]

Question 4a

The following piecewise function models the depth of a pool, in metres.

$$d(x) = \begin{cases} \frac{1}{2}(x+a)^2 + b, & 0 \leq x \leq 2 \\ -2, & 2 < x \leq 10 \\ \frac{1}{4}x + c, & 10 < x \leq d \end{cases}$$

where x represents the horizontal distance in metres from the deep end and a, b and c are constants. The depth at both ends of the pool is 0 m.

(a) Find the values of a, b and c such that f is a continuous function.

[5 marks]

Question 4b

(b) Find the value of d.

[1 mark]

Question 4c

(c) Write down the maximum depth of the pool and the length of the pool.

[1 mark]

Question 5a

A basketball streaming site offers three membership options depending on the number of matches the member wants to watch per week. However, if the member does want to watch more matches in a given week they will be charged per match. The membership options are summarized in the table below.

Membership	Matches/Week	Extra charge/Match	Weekly Cost
Casual	7	\$0.75	\$6.50
Baller	14	\$0.50	\$8.50
Premium	Unlimited	-	\$11.50

(a) The total weekly cost, $\$C$, for the casual and baller memberships can be modelled as a piecewise linear function, where m is the number of matches watched in a given week. Determine the models for each type of membership.

[4 marks]

Question 5b

(b) Find the total weekly cost for each membership if in a given week a member wants to watch 20 games.

[3 marks]

Question 6a

The depth, D m, of an underwater sound wave can be modelled by the function

$$D(t) = 18 - 3.4\sin(0.523t)$$

where t is the elapsed time, in seconds, since the first sound wave was detected by the sensor.

(a) Find the minimum and maximum depths of the sound waves as they pass the sensor.

[3 marks]

Question 6b

(b) Find the first time after 12 seconds at which the depth of the wave reaches 18.2 m.

[3 marks]

Question 7a

The income tax rates for a country are shown in the table below.

Income, $\$x$	Income tax rate, $y\%$
$0 < x \leq 22\,000$	0
$22\,000 < x \leq 64\,000$	18
$64\,000 < x \leq 130\,000$	22
$x > 130\,000$	29

(a) Calculate the amount of tax payable on the first $65 000 of income.

[2 marks]

Question 7b

(b) Calculate the income of someone who has $11 520 of income tax payable.

[3 marks]

Question 7c

John is paid an annual salary, before tax, of $75 000. He works 10 months of the year and then he decides to take the rest of the year off.

(c) Calculate the amount of tax payable for John.

[3 marks]

Question 8a

A Ferris wheel rotates at a constant speed and the height of a particular seat above the ground is modelled by the function

$$H(t) = a\sin(bt - c) + d, \qquad 0 \le t \le 48$$

where H is the height of the seat above the ground, in metres, and t is the elapsed time, in seconds, since the start of the ride.

The ride starts from the lowest point on the Ferris wheel and takes a total of 48 seconds.

(a) Find the value of b.

[2 marks]

Question 8b

The seat reaches a minimum height of 12 m and a maximum height of 42 m.

(b) Find the values of a, c and d.

[3 marks]

Question 8c

Passengers on the Ferris wheel have the best view when their seat is above 25 m.

(c) Calculate the number of seconds for which the passengers have the best view.

[3 marks]

Question 9a

A company producing small boats sells 60 boats per month for a sale price of \$1200, with each boat costing \$700 to produce. Reliable market research suggests that for each increase (or decrease) of the sale price by \$50 the company will sell 10 units less (or more).

(a) Given that N is the number of boats the company can sell per month with a sale price of $\$x$, show that $N(x) = -\frac{1}{5}x + 300$.

[2 marks]

Question 9b

(b) Given that P is the total monthly profit the company makes from selling the boats for a sale price of $\$x$, show that $P(x) = -\frac{1}{5}x^2 + 440x - 210\,000$.

[3 marks]

Question 9c

(c) Find the number of boats the company must produce to maximise monthly profit, given that the maximum monthly production is

(i) 75 boats

(ii) 115 boats.

[3 marks]

Question 9d

(d) Write down two intervals of x for which the company makes a loss and state an economic reason why for each interval.

[1 mark]

Question 10a

For a particular type of coffee, a typical mug contains 100 mg of caffeine. The half-life of the amount of caffeine in the bloodstream is 2.5 hours.

Assuming the 100 mg of caffeine from a mug of coffee is absorbed immediately after drinking it, the amount of caffeine, C mg, left in the bloodstream t hours after consumption can be modelled by the equation

$$C = Ae^{-kt}$$

where A and k are positive constants.

a)
Write down the value of A.

[2 marks]

Question 10b

(b)
Find the value of k, giving your answer to 2 significant figures.

[2 marks]

Question 10c

(c)
Find the amount of caffeine in the bloodstream after 6 hours.

[3 marks]

Question 10d

A consumer wishes to cut down their caffeine intake and so makes a drink using half the amount of coffee in a mug.

d)
Find the amount of caffeine in the bloodstream for this consumer after 6 hours and state an assumption you have made in finding your answer.

[3 marks]

Question 11a

A rhino is raised in a zoo and his height, h metres, is modelled by the logistic function

$$h(t) = \frac{L}{1 + 1.9e^{-0.27t}}, \qquad t \geq 0,$$

where t is the number of years since his birth. The rhino's height reaches a limit of 1.82 m as he ages.

(a)
State the value of L.

[2 marks]

Question 11b

(b)
Find the rhino's height on his 12th birthday.

[2 marks]

Question 11c

The rhino's nth birthday is the first birthday in which he is double the value of $h(0)$.

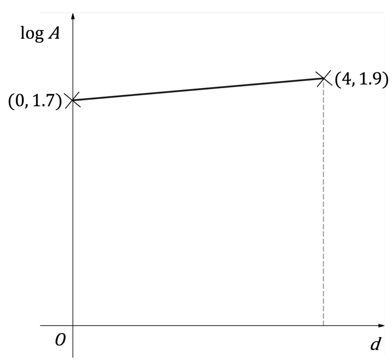
(c)
Find the value of n.

[3 marks]

Question 12a

The surface of a large pond is partially covered by algae. A limnologist (a scientist who studies freshwater systems) monitors the area, A m^2, of the pond covered by algae, d days, after first discovering its presence.

The limnologist plots a graph of $\log A$ against d, and after 4 days the graph is a straight line passing through the points (0, 1.7) and (4, 1.9).



The limnologist believes the area of the pond covered by algae can be modelled by the equation $A = A_0 b^d$.

a)
Find the value of A_0, giving your answer to two significant figures, and explain its meaning in the context of the algae covering the lake.

[2 marks]

Question 12b

b)

i)

Find the gradient of the straight line.

ii)

Hence find the value of b correct to 2 significant figures

[2 marks]

Question 12c

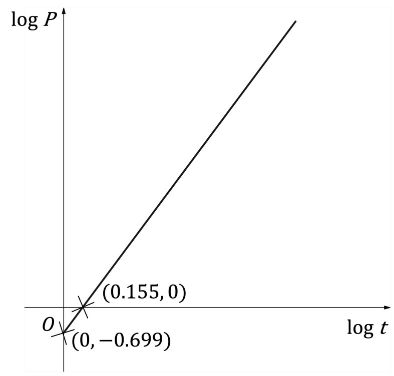
c)

Using the rounded values for A_0 and b in the model predict the area of the pond covered by algae after 20 days.

[2 marks]

Question 13a

In a production process the amount of a pollutant, P ppm (parts per million), in the surrounding air seconds after the process began, is monitored. A chemist produces the graph below of the first 10 seconds of the process.



The graph passes through the points $(0, -0,7)$ and $(1,4,0)$. The chemist suggests that a model of the form $P = at^b$, where a and b are constants, can be used to predict the amount of pollutant in the air.

a)
Find the gradient of the graph.

[2 marks]

Question 13b

b)
Find the equation of the straight line.

[1 mark]

Question 13c

c)
Find the values of a and b.

[3 marks]

Question 13d

The process stops after a maximum running time of 20 seconds.

d)
Find the maximum amount of the pollutant produced during one occurrence of the production process. State your answer to 2 significant figures.

[2 marks]