3.10 Graph Theory

Question Paper

Course	DP IB Maths
Section	3. Geometry & Trigonometry
Topic	3.10 Graph Theory
Difficulty	Very Hard

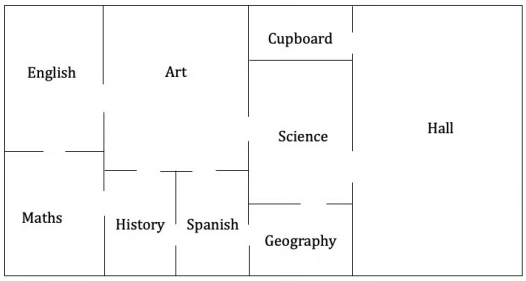
Time allowed: 90

Score: /73

Percentage: /100

Question 1a

The diagram below shows the plan of a school building.



a)
Construct a graph to represent this information, using vertices to represent rooms and edges to represent the connecting doors.

[2 marks]

Question 1b

For a prank, a student releases a monkey into the English classroom at 8:30 pm on Thursday.

b)
Given that the monkey wanders through the rooms at random, find the probability that the 7^{th} room it wanders into (after leaving the English classroom) is the Geography classroom.

[3 marks]

Question 1c

The monkey continues to wander at random through the school building all night until the cleaner arrives at 6 am on Friday morning

c)
By inspecting the steady state probabilities, write down the room in which the cleaner is most likely to discover the monkey.

[2 marks]

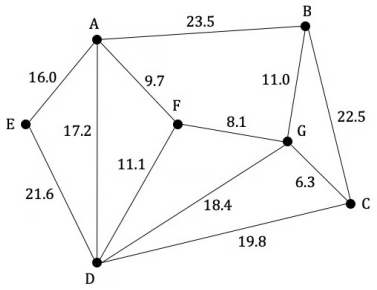
Question 1d

d)
Stating clearly any assumptions you have made, find an approximation for the total length of time the monkey is likely to have spent in the Maths classroom before the cleaner arrives.

[3 marks]

Question 2a

Consider the weighted graph G below. Vertex A represents a water source and the remaining vertices represent water features in a park. The edges represent pipes that could be installed to connect the features to the water source, with the weightings being their lengths in metres.



a)
Explain, with a reason, which of the two relevant algorithms would be most efficient for finding the minimum length of pipe needed to connect all of the features to the water supply.

[2 marks]

Question 2b

a)
Find the minimum length of pipe required to connect all of the water features to the water source using the algorithm stated in part (a). Show each step of the algorithm clearly.

[4 marks]

Question 2c

After the minimum-length system from part (b) has been installed, a fault with the feature at vertex G is discovered such that the feature and all pipes directly connected to it must be disconnected from the water system. Any features that had been connected to the system via vertex G must now be re-connected, using new pipes, by the shortest possible route that does not go through vertex G.

c)
Given that the cost of the piping is £3.80 per metre, find the amount of money that would have been saved if the system was initially designed without the feature at vertex G.

[6 marks]

Question 3a

Min Son is installing 8 lamps in her garden and wishes to connect them together so that she will ultimately only need to connect one of the lamps directly to the electrical supply. Each lamp is set in a hole and the electrical cables run underground from the base of each lamp.

Each vertex listed in the table below represents a lamp and the weighting of each edge is the horizontal distance in metres between each pair of lamps.

	A	B	C	D	E	F	G	H
A	-	7	10.2	10.5	15.4	13.1	9.3	6.5
B	7	-	6.5	9.3	13.1	15.4	10.5	10.2
C	10.2	6.5	-	3.8	6.7	11.3	6.8	8.9
D	10.5	9.3	3.8	-	4.9	7.5	3.5	6.7
E	15.4	13.1	6.7	4.9	-	9.2	7.5	11.3
F	13.1	15.4	11.3	7.5	9.2	-	4.9	6.7
G	9.3	10.5	6.8	3.5	7.5	4.9	-	3.8
H	6.5	10.2	8.9	6.7	11.3	6.7	3.8	-

a)
Find the minimum length of electrical cable required to connect all of the lamps together.

[4 marks]

Question 3b

b)
State two assumptions that have been made.

[2 marks]

Question 4

In a town there are 8 local attractions and each attraction has its own website that it maintains. Some of the websites contain links to the other websites as shown in the adjacency table below.

	A	B	C	D	E	F	G	H
A	0	1	0	1	0	0	0	1
B	0	0	0	0	0	0	1	0
C	0	0	0	1	0	1	0	0
D	0	0	1	0	0	0	0	0
E	0	1	0	1	0	1	0	0
F	0	0	1	0	1	0	0	0
G	1	1	0	0	1	1	0	1
H	1	0	0	0	0	0	0	0

On its results page, the search engine Beegle ranks the websites in the following order from highest to lowest: C, E, H, F, D, G, A, B.

Beegle states that it ranks websites based on the relative number of clicks each one receives, with those pages that are clicked on more often ranking higher in the search results.

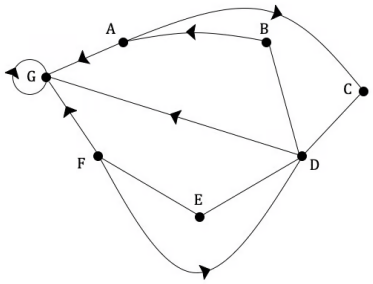
An experiment is conducted to test Beegle's claim, whereby a volunteer randomly clicks on the links between the different websites 1000 times.

By investigating the probabilities of clicking on the different websites, determine whether Beegle's statement is likely to be true. State any assumptions you make in reaching your conclusion.

[5 marks]

Question 5a

Consider the graph R below.



a)
Find the number of walks of length 4 or less that start at vertex D and finish at vertex G.

[3 marks]

Question 5b

b)
Determine the starting position of the random walk of length 7 that is least likely to finish at G.

[3 marks]

Question 5c

c)
Without doing any further calculations, write down the steady state probabilities for the graph. Be sure to justify your answer.

[1 mark]

Question 5d

The directional arrow on the edge AG is removed, so that edge AG is now traversable in both directions.

d)
For a random walk with a very large number of steps, determine the order, from most to least likely, of the vertices upon which the walk is likely to finish.

[3 marks]

Question 6a

The table below shows the length of track, in kilometres, between 8 local train stations.

	A	B	C	D	E	F	G	H
A	-							
B	9	-						
C	7	-	-					
D	-	2	16	-				
E	-	-	5	-	-			
F	-	-	8	-	6	-		
G	18	-	-	-	19	-	-	
H	6	-	15	-	-	21	-	-

A railway engineer must walk all the lengths of track between the stations, in either direction, in order to inspect them. Whilst inspecting track it takes the engineer an average of 1.8 minutes to walk a distance of 100 metres, but if she is walking back over track she has already inspected she walks at a rate of 4 km/h.

a)
Explain why the inspector cannot return to the same station she starts at without walking along some lengths of track more than once.

[1 mark]

Question 6b

b)
Find the least amount of time that the inspector requires to inspect all of the lengths of track if she starts and finishes at station A.

[5 marks]

Question 6c

c)
Given the context of the question, explain why the solution in (b) is not optimal.

[1 mark]

Question 7a

The table of least weights for a complete graph of 5 vertices is shown below. Each vertex in the table represents a house that Angelina must visit to make a delivery. The weightings represent the time taken, in minutes, to travel between the houses. Angelina wants to find the quickest route she can take that visits all of the houses exactly once.

	A	B	C	D	E
A	-	11	16	11	7
B	11	-	6	12	5
C	16	6	-	9	11
D	11	12	9	-	8
E	7	5	11	8	-

a)
By deleting each vertex in turn, find a best lower bound for the shortest route Angelina can take.

[4 marks]

Question 7b

b)
Starting at vertex A and using the nearest neighbour algorithm, find an upper bound for the shortest route Angelina can take.

[4 marks]

Question 7c

c)
Hence determine the total time of the shortest route that Angelina can take. Give a reason for your answer.

[1 mark]

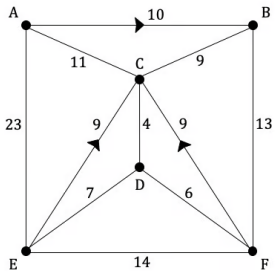
Question 7d

d)
Write down a possible route of shortest length, given that Angelina starts and finishes at house A.

[1 mark]

Question 8a

The graph below shows 6 treasures (vertices) hidden in a maze and the tunnels (edges) that connect them. The weights on the edges indicate the lengths of the tunnels in metres. Some of the tunnels can only be travelled along in one direction as indicated on the graph.



a)
Construct a table showing the shortest distance between each pair of treasures.

[5 marks]

Question 8b

Ankunda wants to find the shortest distance he will have to travel to collect each of the treasures, starting and finishing at the same location.

b)
Find a best upper bound for the shortest route Ankunda can take, given that
i)
he must start and finish at the location of treasure C
ii)
he can start and finish at the location of any treasure.

[8 marks]