5.2 Further Differentiation

Question Paper

Course	DP IB Maths
Section	5. Calculus
Topic	5.2 Further Differentiation
Difficulty	Hard

Time allowed: 110

Score: /91

Percentage: /100

Question 1a

(a) Use the product rule to find the derivative of $f(x) = (3x - 7)(4 - 2x^2)$

[2 marks]

Question 1b

(b) Use the quotient rule to find the derivative of $g(x) = \frac{-7x}{x^3 - 1}$

[3 marks]

Question 1c

(c) Use the chain rule to find the derivative of $h(x) = (5 - 3x)^5$

[2 marks]

Question 1d

(d) Find the derivative of $j(x) = \frac{3x^{\frac{1}{5}}}{1 - x^{\frac{2}{3}}}$

[4 marks]

Question 2a

Find an expression for the derivative of each of the following functions:

(a) $f(x) = \mathrm{e}^{3x} \tan x$

[2 marks]

Question 2b

(b) $g(x) = \sin(3x^2 + 5)$

[2 marks]

Question 2c

(c) $h(x) = \dfrac{-\cos^2 x}{\ln x}$

[3 marks]

Question 2d

(d) $j(x) = x\sqrt{x} - \frac{1}{\sqrt[3]{x}}$

[3 marks]

Question 3

Consider the function f defined by $f(x) = 2x + \cos^3 x,\ x \in \mathbb{R}$.

By considering the derivative of the function, show that f is increasing everywhere on its domain.

[5 marks]

Question 4a

Consider the function g defined by $g(x) = e^x - 7x,\ x \in \mathbb{R}$.

(a) Show that the equation of the tangent to the graph of g at $x = \ln 3$ may be written in the form $y = -4x - 3(\ln 3 - 1)$.

[5 marks]

Question 4b

(b) By considering $g'(x)$ show that there is a point on the graph of g at which the normal to the graph is vertical, and determine the exact coordinates of that point.

[3 marks]

Question 5a

Consider the function h defined by $h(x) = \cos x - e^{2x} \sin x,\ x \in \mathbb{R}$.

(a) Find an expression for $h'(x)$.

[3 marks]

Question 5b

(b) Hence determine an equation for the tangent to the graph of h at $x = \pi$.

[4 marks]

Question 6

Let $f(x) = g(x)h(x)$, where g and h are functions such that $g(x) = 3x^2h(x)$ for all $x \in \mathbb{R}$.

Given that $h(-1) = 2$ and $h'(-1) = -2$, find the equation of the tangent to the graph of f at $x = -1$.

[7 marks]

Question 7a

Consider the curve with equation $y = e^{x^3}$, defined for all values of $x \in \mathbb{R}$.

(a) Find an expression for $\frac{d^2y}{dx^2}$.

[5 marks]

Question 7b

(b) Hence determine the values of x for which the curve is

(i) concave up

(ii) concave down.

Your answers should be given as exact values.

[4 marks]

Question 7c

(c) Use your answer to part (b) to show that the curve has two points of inflection, and determine the exact values of their coordinates.

[4 marks]

Question 8a

Consider the function f defined by $f(x) = xe^{3\cos x}$, for $-\pi \leq x \leq \pi$.

(a) Find the number of points at which the graph of f has a horizontal tangent.

[1 mark]

Question 8b

The point A is the point on the graph of f for which the x-coordinate is $\frac{\pi}{2}$.

(b) Show algebraically that the gradient of the tangent to the graph of f at point A is $\frac{2-3\pi}{2}$.

[4 marks]

Question 8c

(c) Hence find the equation of the normal line to the graph of f at point A, and determine where that line intersects the x-axis.

[5 marks]

Question 8d

(d) Show algebraically that the graph of f intersects the line $y = x$ in exactly three places, and determine the coordinates of the points of intersection.

[4 marks]

Question 9

Let $f(x) = \frac{\sqrt{3}}{2}\cos 2x$ and $g(x) = \frac{1}{2}\sin 2x$, for $0 \le x \le \pi$.

Solve the equation $f'(x) = g'(x)$, giving your answers as exact values.

[5 marks]

Question 10

An ice sculptor has created an abstract minimalist ice sculpture in the shape of a cylinder with radius r m and height $8r$ m. The sculpture is of solid ice throughout.

After a power cut that shuts off the sculptor's freezer, the sculpture begins melting such that the volume of ice is decreasing at a constant rate of $0.4\ \text{m}^3$ per hour.

Assuming that while it melts the sculpture remains at all times in the shape of a cylinder which is mathematically similar to the original cylinder, find the rate at which the sculpture's surface area is changing at the point when its radius is 0.3 m.

[6 marks]

Question 11

A hemispherical bowl is supported with its curved surface on the bottom, such that the plane defined by the open top of the bowl is at all times horizontal. The bowl contains liquid, with the volume of liquid in the bowl being given by the formula

$$V = \frac{1}{3}\pi h^2(3r - h)$$

where r is the radius of the bowl and h is the depth of liquid (i.e., the height between the bottom of the bowl and the surface level of the liquid).

The bowl is leaking liquid through a small hole in its bottom at a rate directly proportional to the depth of liquid.

Show that the rate of change of the depth of liquid in the bowl is

$$-\frac{k}{\pi(2r - h)}$$

where k is a positive constant.

[5 marks]