5.6 Differential Equations

Question Paper

Course	DP IB Maths
Section	5. Calculus
Topic	5.6 Differential Equations
Difficulty	Medium

Time allowed: 90

Score: /68

Percentage: /100

Question 1a

Consider the first-order differential equation

$$\frac{dy}{dx} - 5x^4 = 3$$

(a) Find the general solution to the differential equation, giving your answer in the form $y = f(x)$.

[3 marks]

Question 1b

(b) Find the specific solution to the equation given that $y = 40$ when $x = 2$.

[2 marks]

Question 2a

Use separation of variables to find the general solution of each of the following differential equations, giving your answers in the form $y = f(x)$:

(a)

$$\frac{dy}{dx} = \frac{4x^2}{y^4}$$

[4 marks]

Question 2b

(b)

$$\frac{dy}{dx} = (x^2 + 1)e^{-y}$$

[4 marks]

Question 3a

Use separation of variables to solve each of the following differential equations for which satisfies the given boundary condition:

a)

$$\frac{dy}{dx} = xy^2; \quad y(2) = 1$$

[5 marks]

Question 3b

b)

$$(x+3)\frac{\mathrm{d}y}{\mathrm{d}x}=\sec y;\quad y(-2)=\frac{3\pi}{2}$$

[5 marks]

Question 4a

Scientists are studying a large pond where an invasive plant has been observed growing, and they have begun measuring the area, A m^2, of the pond's surface that is covered by the plant. According to the scientists' model, the rate of change of the area of the pond covered by the plant at any time, t, is proportional to the square root of the area already covered.

(a) Write down a differential equation to represent the scientists' model.

[2 marks]

Question 4b

(b) Solve the differential equation to show that

$$A = \left(\frac{kt + c}{2}\right)^2$$

where k is the constant of proportionality and c is a constant of integration.

[4 marks]

Question 4c

At the time when the scientists begin studying the pond the invasive plant covers an area of 100 m^2. One week later the area has increased to 225 m^2.

(c) Use this information to determine the values of k and c.

[3 marks]

Question 4d

The pond has a total area of 250 000 m^2.

(d) Determine how long it will take, according to the scientists' model, for the invasive plant to cover the entire surface of the pond.

[2 marks]

Question 5a

At any point in time, the rate of growth of a colony of bacteria is proportional to the current population size. At time $t = 0$ hours, the population size is 5000.

a)
Write a differential equation to model the size of the population of bacteria.

[1 mark]

Question 5b

After 1 hour, the population has grown to 7000.

b)
By first solving the differential equation from part (a), determine the constant of proportionality.

[6 marks]

Question 5c

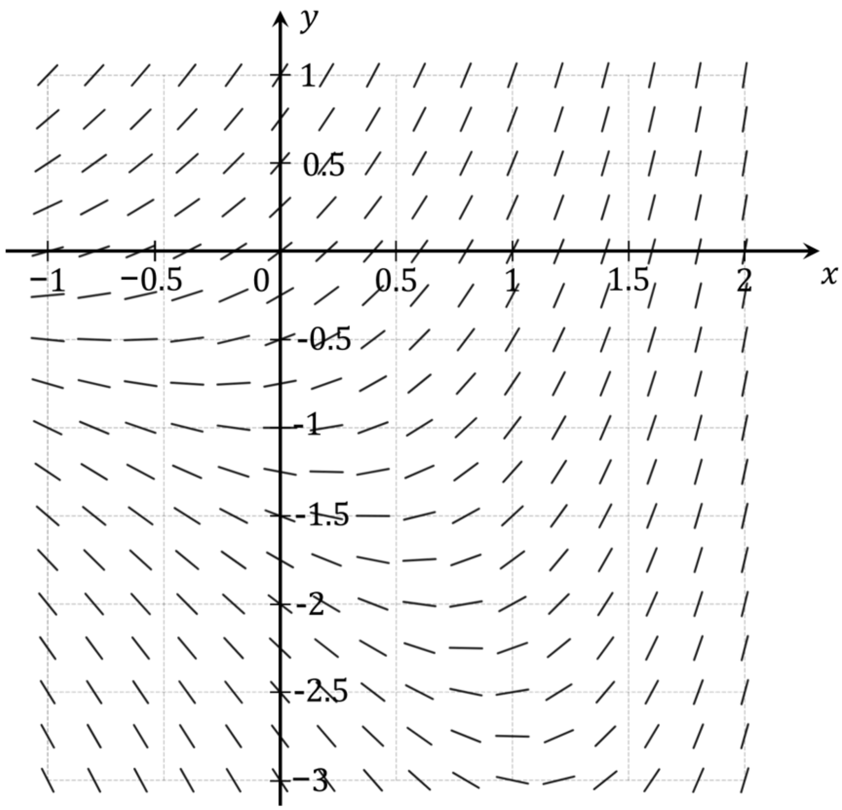
c)
(i)
Show that, according to the model, it will take exactly $\frac{\ln 20}{\ln 7 - \ln 5}$ hours (from $t = 0$) for the population of bacteria to grow to $100\ 000$.

(ii)
Confirm your answer to part (c)(i) graphically.

[5 marks]

Question 6a

The graph below shows the slope field for the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = e^x + y$ in the intervals $-1 \le x \le 2$ and $-3 \le y \le 1$.



(a) Calculate the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at the point $(0, -3)$.

[1 mark]

Question 6b

(b) On the graph above sketch:

(i) a curve that represents the points where $\frac{dy}{dx} = 0$

(ii) the solution curve that passes through the point $(0, -1)$

(iii) the solution curve that passes through the point $(0, -2)$

[6 marks]

Question 7a

Consider the differential equation

$$\frac{dy}{dx} = \frac{y}{x} + 1$$

with the boundary condition $y(1) = 0$.

(a) Apply Euler's method with a step size of $h = 0.2$ to approximate the solution to the differential equation at $x = 2$.

[3 marks]

Question 7b

(b) It can be shown that the exact solution to the differential equation with the given boundary condition is $y = x \ln x$. Compare your approximation from part (a) to the exact value of the solution at $x = 2$.

[3 marks]

Question 7c

(c) Explain how the accuracy of the approximation in part (a) could be improved.

[1 mark]

Question 8a

A particle moves in a straight line, such that its displacement x at time t is described by the differential equation

$$\dot{x} = \frac{te^{3t^2} + 1}{4x^2}, \quad t \geq 0$$

At time $t = 0$, $x = \frac{1}{2}$.

(a) By using Euler's method with a step length of 0.1, find an approximate value for x at time $t = 0.3$.

[3 marks]

Question 8b

(b) (i) Solve the differential equation with the given boundary condition to show that

$$x = \frac{1}{2}\sqrt[3]{e^{3t^2} + 6t}$$

(ii) Hence find the percentage error in your approximation for x at time $t = 0.3$.

[5 marks]