1.9 Further Complex Numbers

Question Paper

Course	DP IB Maths
Section	1. Number & Algebra
Topic	1.9 Further Complex Numbers
Difficulty	Medium

Time allowed: 110

Score: /90

Percentage: /100

Question 1a

Consider $w = \frac{z_1}{z_2}$, where $z_1 = 2 + 2\sqrt{3}\,\mathrm{i}$ and $z_2 = 2 + 2\mathrm{i}$.

a)
Express w in the form $w = a + b\mathrm{i}$.

[3 marks]

Question 1b

b)
Write the complex numbers z_1 and z_2 in the form $re^{\mathrm{i}\theta},\ r \geq 0,\ -\pi < \theta < \pi$.

[4 marks]

Question 1c

c)
Express w in the form $re^{\mathrm{i}\theta},\ r \geq 0,\ -\pi < \theta < \pi$.

[3 marks]

Question 2

Solve the equation $z^3 = 27\mathrm{i}$, giving your answers in the form $a + b\mathrm{i}$.

[6 marks]

Question 3a

Let $z_1 = 6\mathrm{cis}(\frac{\pi}{6})$ and $z_2 = 3\sqrt{2}e^{\mathrm{i}\left(\frac{\pi}{4}\right)}$.

a)
Giving your answers in the form $r\mathrm{cis}\theta$, find

(i)
$z_1 z_2$

(ii)
$\frac{z_1}{z_2}$.

[4 marks]

Question 3b

b)
Write z_1 and z_2 in the form $a + b\text{i}$.

[2 marks]

Question 3c

c)
Find $z_1 + z_2$, giving your answer in the form $a + b\text{i}$.

[2 marks]

Question 3d

It is given that z_1^* and z_2^* are the complex conjugates of z_1 and z_2 respectively.

d)
Find $z_1^* + z_2^*$, giving your answer in the form $a + b\text{i}$.

[2 marks]

Question 4a

Let $z_1 = 2\text{cis}(\frac{\pi}{3})$ and $z_2 = 2 + 2\text{i}$.

a)
Express

(i)
z_1 in the form $a + b\text{i}$

(ii)
z_2 in the form $r\text{cis}\theta$

[2 marks]

Question 4b

b)
Find $w_1 = z_1 + z_2$, giving your answer in the form $a + b\text{i}$.

[2 marks]

Question 4c

c)
Find $w_2 = z_1 z_2$, giving your answer in the form $r\text{cis}\theta$.

[3 marks]

Question 4d

d)
Sketch w_1 and w_2 on a single Argand diagram.

[2 marks]

Question 5a

It is given that that $z_1 = 2e^{i\left(\frac{\pi}{3}\right)}$ and $z_2 = 3\,\text{cis}\left(\frac{n\pi}{12}\right)$, $n \in \mathbb{Z}^+$.

a)
Find the value of $z_1 z_2$ for $n = 3$.

[3 marks]

Question 5b

b)
Find the least value of n such that $z_1 z_2 \in \mathbb{R}^+$.

[3 marks]

Question 6a

Consider the complex number $w = \frac{z_1}{z_2}$ where $z_1 = 3 - \sqrt{3}\,\mathrm{i}$ and $z_2 = 2cis\left(\frac{2\pi}{3}\right)$.

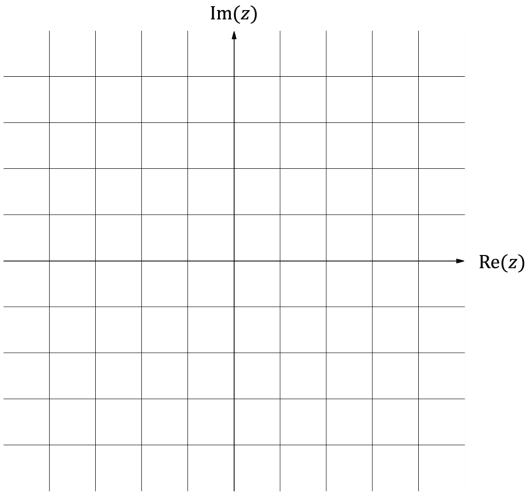
a)
Express w in the form $r\,cis\theta$.

[5 marks]

Question 6b

b)

Sketch z_1, z_2 and w on the Argand diagram below.



[3 marks]

Question 6c

c)

Find the smallest positive integer value of n such that w^n is a real number.

[2 marks]

Question 7a

Consider the complex number $z = -1 + \sqrt{3}i$.

(a)

Express z in the form $r\,\text{cis}\,\theta$, where $r > 0$ and $-\pi < \theta \leq \pi$.

[4 marks]

Question 7b

(b)

Find the three roots of the equation $z^3 = -1 + \sqrt{3}\mathrm{i}$, expressing your answers in the form $r\,\mathrm{cis}\,\theta$, where $r > 0$ and $-\pi < \theta \leq \pi$.

[4 marks]

Question 8a

Consider the equation $z^4 - 1 = 15$, where $z \in \mathbb{C}$.

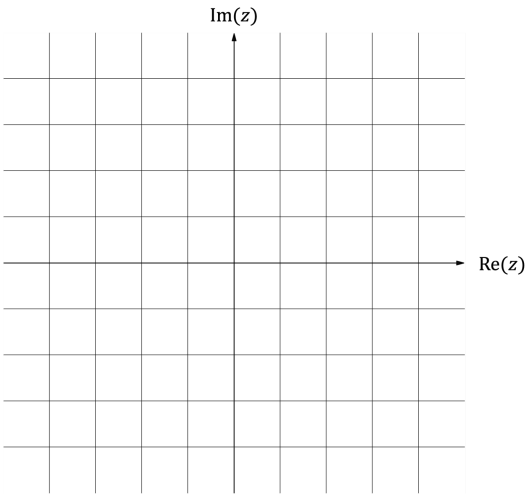
(a)

Find the four distinct roots of the equation, giving your answers in the form $a + b\mathrm{i}$, where $a, b \in \mathbb{R}$.

[4 marks]

Question 8b

(b)
Represent the roots found in part (a) on the Argand diagram below.



[2 marks]

Question 8c

(c)
Find the area of the polygon whose vertices are represented by the four roots on the Argand diagram.

[2 marks]

Question 9a

Consider the complex numbers $w = 3\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$ and $z = 3 - \sqrt{3}i$.

(a)
Write w and z in the form $r\,\text{cis}\,\theta$, where $r > 0$ and $-\pi < \theta \leq \pi$.

[4 marks]

Question 9b

(b)
Find the modulus and argument of zw.

[2 marks]

Question 9c

(c)
Write down the value of zw.

[2 marks]

Question 10a

Let $z = 12 + 16\mathrm{i}$, where $a, b \in \mathbb{R}$.

a)
Verify that $4 + 2\mathrm{i}$ and $-4 - 2\mathrm{i}$ are the second roots of z.

[2 marks]

Question 10b

b)
Hence, or otherwise, find two distinct roots of the equation $w^2 + 4w + (1 - 4\mathrm{i}) = 0$, where $w \in \mathbb{C}$. Give your answer in the form $a + b\mathrm{i}$, where $a, b \in \mathbb{R}$.

[4 marks]

Question 11a

The complex numbers $\omega_1 = 3$ and $\omega_2 = 2 - 2\mathrm{i}$ are roots of the cubic equation $\omega^3 + p\omega^2 + q\omega + r = 0$, where $p, q, r \in \mathbb{R}$.

a)
Write down the third root, w_3, of the equation.

[1 mark]

Question 11b

b)
Find the values of p, q and r.

[4 marks]

Question 11c

c)

Express w_1, w_2 and w_3 in the form $r\,cis\theta$.

[4 marks]