# 5.5 Optimisation

## Question Paper

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| Course | DP IB Maths |
| Section | 5. Calculus |
| Topic | 5.5 Optimisation |
| Difficulty | Hard |

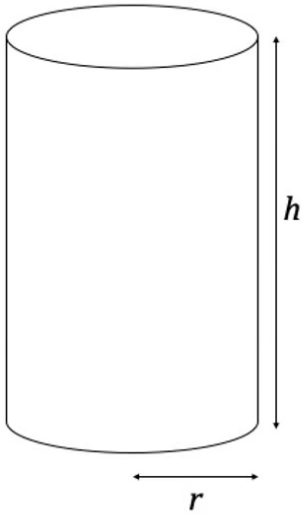
**Time allowed:** 140

**Score:** /110

**Percentage:** /100

## Question 1a

A cylinder with height $h$ and radius $r$ is shown in the diagram below.




The sum of the diameter and height for this cylinder is 20 cm.

(a) Write down an equation for the total surface area of the cylinder, $A$, in terms of $r$.

**[2 marks]**

## Question 1b

(b) Find $\frac{dA}{dr}$.

**[2 marks]**

## Question 1c

(c) Find the value of r when the surface area is maximised.

**[2 marks]**

## Question 2a

The population of rabbits in a colony is modelled over the course of a year by the function

$$f(t) = -\frac{1}{8}t^3 + \frac{11}{5}t^2 + 20,\ 0 \leq t \leq 12,$$

where $t$ is the time from the start of the year measured in months.

(a) Find the population of rabbits in the colony at $t = 8$.

**[2 marks]**

## Question 2b

(b) Find the maximum population of the colony and the time at which it occurs.

**[3 marks]**

## Question 2c

(c) Find the value of $t$ for which the population of rabbits is increasing most rapidly.

**[2 marks]**

## Question 3a

Daphne is starting a business selling fabric via an online retail website. The amount of fabric, in metres, that she thinks she will sell in one month, F, can be modelled by

$$F = 151 - 5m$$

where $m$ is the price per metre in US dollars (USD).

Daphne will incur various costs through running her business, both fixed and variable. The fixed costs total \$89 per month and the variable costs can be expressed by $3m$.

(a) Write down an expression for the monthly profit, $P$, in terms of $m$.

**[3 marks]**

## Question 3b

(b) Find the price per metre, $m$, that will give Daphne the maximum monthly profit.

**[3 marks]**

### Question 3c

In her first month of trading, Daphne must not let her costs exceed \$128.

(c) Find the greatest profit that Daphne is able to make in the first month.

**[3 marks]**

### Question 4a

Patroclus, a would-be Olympic javelin thrower, throws a javelin during a training session. The height of the javelin's point can be modelled by the equation

$$h(t) = 1.75 + 20.2t - 4.90t^2$$

where $t$ is the time, in seconds, that has passed since the javelin was released, and $h(t)$ is the height of the javelin above the ground, in metres.

(a) Find $h'(t)$.

**[2 marks]**

### Question 4b

(b) (i) Find the stationary point for $h(t)$.

(ii) Justify that the stationary point is a maximum point.

**[6 marks]**

### Question 4c

(c) Find the greatest vertical distance that the javelin's point travels above the height from which it was released.

**[1 mark]**

## Question 5a

A petrol tanker transports fuel in a cylindrical container with length $l$ metres, where $l = 5r + 3$. The cylindrical container sits on top of a truck bed and cannot exceed its width of 2.6 m.

(a) Show that the volume, $V$, of the container can be expressed as $V(l) = \frac{\pi l(l-3)^2}{25}$.

**[2 marks]**

## Question 5b

(b) State the domain of the function $V(l)$.

**[2 marks]**

## Question 5c

(c) Find the maximum volume of the container.

**[2 marks]**

## Question 5d

The transportation company decides to use containers with the length used in part (c). They find that their costs $C(x)$, in dollars, vary depending on the size of the order and can be modelled by the function

$$C(x) = \frac{7}{8}(x-5)^3 - 28x + 390$$

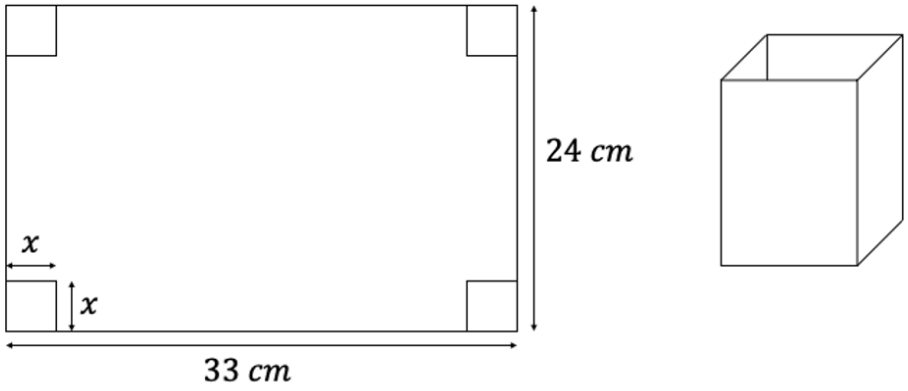
where $x$ is the number of cubic metres of empty space within the container.

d) Find the percentage to which the container should be filled at which the costs are at a minimum.

**[3 marks]**

## Question 6a

Percy has been given a sheet of metal of length 24 cm and width 33 cm. He has been asked to use this material to design an open box that will be used to hold stationery. He removes a square of length $x$ from each corner of the sheet of metal and the remainder of the sides are then folded up to create the box as shown in the diagrams below.




(a) Write down the interval for the possible values of $x$.

**[2 marks]**

### Question 6b

(b) Show that the volume of the box, $V$, can be expressed as

$$V(x) = 4x^3 - 114x^2 + 792x$$

**[3 marks]**

### Question 6c

(c) Find $V'(x)$ .

**[2 marks]**

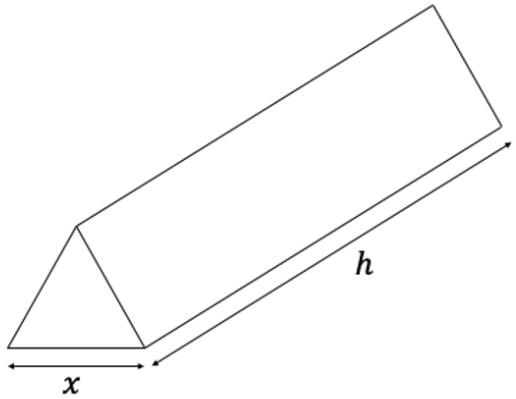
### Question 6d

(d) Using your answer from part (c), find the value of $x$ that maximises the volume of the box.

**[2 marks]**

## Question 7a

A chocolate manufacturer wishes to introduce a new chocolate product in the shape of an equilateral triangular prism. The edge of each triangular face has length $x$ cm and the height of the prism is $h$ cm, as shown in the diagram below. The sum of the height of the prism and all three triangular edges is 18 cm.




(a) Explain why $x \neq 6$.

**[1 mark]**

## Question 7b

(b) Show that the volume of the triangular prism, $V$, can be described by

$$V = \frac{3\sqrt{3}}{2}\left(3x^2 - \frac{1}{2}x^3\right)$$

**[4 marks]**

## Question 7c

(c) By differentiating the equation in part (b), find the length of $x$ for which the volume is a maximum.

**[4 marks]**

## Question 7d

(d) Find the amount of packaging required when the volume of the box is at a maximum.

**[3 marks]**

## Question 8a

Check, Mate! is a company that produces luxury chess sets for discerning chess set connoisseurs. The company's profits $P(x)$, in thousands of UK pounds (£1000), can be modelled by the function

$$P(x) = 0.32x^3 - 12.4x^2 + 150x - 480$$

where $x$ is the number of chess sets (in hundreds) sold per year. Because of manufacturing constraints, the maximum number of chess sets that the company can sell in a year is 2500.

(a) (i) State why there is no need to consider values of $x$ greater than 25.

(ii) Sketch a graph of $P(x)$ for $0 \leq x \leq 25$.

**[3 marks]**

## Question 8b

(b) (i) Find the stationary points on the graph, and the numbers of chess sets sold and profits that correspond to those points.

(ii) Find the maximum profit that the company can make in a year, and the number of chess sets the company must sell to make that profit.

**[5 marks]**

## Question 8c

(c) Calculate

(i) the average rate of change of $P(x)$ between $x = 5$ and $x = 6$

(ii) the instantaneous rate of change of $P(x)$ at $x = 5$.

In each case include the units, and explain the meaning of the value you find.

**[5 marks]**

## Question 8d

(d) State the values of $x$ for which the instantaneous rate of change of $P(x)$ is negative.
Explain the meaning of this result.

**[3 marks]**

## Question 9a

A manufacturing company is producing tins that must have a capacity of $470\ \text{cm}^3$. The tins are in the shape of a cylinder with a height of $h$ cm and a base radius of $r$ cm.

(a) Show that the surface area of the cylinder in $\text{cm}^2$, including the two circular ends, may be written as

$$A = 2\pi r^2 + \frac{940}{r}$$

**[4 marks]**

## Question 9b

(b) Sketch the graph of $A = 2\pi r^2 + \frac{940}{r}$.

**[2 marks]**

## Question 9c

The company would like to minimise the amount of metal used to make the tins.

(c) (i) Find the stationary point on the graph of $A = 2\pi r^2 + \frac{940}{r}$, and justify that it is a minimum point.

(ii) Hence find the minimum possible surface area for the tin, and the base radius that corresponds to that minimum area.

**[5 marks]**

## Question 9d

A commercially available tin of chopped tomatoes on sale in the UK has a capacity of $470\ \text{cm}^3$ and a base radius of 3.7 cm.

(d) Determine the percentage difference between the surface area of that tin of chopped tomatoes and the minimum possible surface area for a tin with the same capacity.

**[3 marks]**

## Question 10

Two numbers, $x$ and $y$, are such that $x > y$ and the difference between the two numbers is 7.

Find the minimum possible value of the product $xy$, and the values of $x$ and $y$ that correspond to that minimum value.

**[6 marks]**

## Question 11a

After analysing several years of company data, a fast food company has determined that the rate of change of its sales figures can be modelled by the equation

$$\frac{\mathrm{d}M}{\mathrm{d}x} = -0.068x^3 + 0.72x^2 - 0.88x - 1.9, \qquad 0 \leq x \leq 10$$

where $M$ represents the number of meals sold in a week (in thousands of meals sold), and $x$ represents the amount spent on advertising during the preceding week (in thousands of euros).

It is known as well that 5988 meals are sold in a week where 2000 euros had been spent on advertising during the preceding week.

(a) Find an expression for $M(x)$.

**[6 marks]**

## Question 11b

(b) Find the maximum number of meals that the company can expect to sell in a week, and the amount of money that the company should spend on advertising during the preceding week to bring about that level of sales. Give your answers to the nearest meal sold and the nearest euro, respectively. Be sure to justify that the value you find is indeed a maximum.

**[7 marks]**