1.7 Matrices

Question Paper

Course	DP IB Maths
Section	1. Number & Algebra
Topic	1.7 Matrices
Difficulty	Hard

Time allowed: 90

Score: /72

Percentage: /100

Question 1a

Consider the following matrices:

$$\boldsymbol{A}=\begin{pmatrix} x & 7 \\ -4 & 3 \end{pmatrix} \qquad \boldsymbol{B}=\begin{pmatrix} 2 & y \\ 4 & 8 \end{pmatrix}$$

$$\boldsymbol{C}=\begin{pmatrix} 1 & a \\ 4 & 8 \end{pmatrix} \qquad \boldsymbol{D}=\begin{pmatrix} b & 2 \\ -9 & -10 \end{pmatrix} \qquad \boldsymbol{E}=\begin{pmatrix} -34 & 62 \\ -66 & -52 \end{pmatrix}$$

$$\boldsymbol{M}=\begin{pmatrix} 3 & 2c \\ -c & 6 \end{pmatrix} \qquad \boldsymbol{N}=\begin{pmatrix} 10 & d+24 \\ d & 22 \end{pmatrix}$$

where $a, b, c, d, x, y \in \mathbb{R}$ are constants.

a)
Given that $\boldsymbol{A}+q\boldsymbol{B}=z\boldsymbol{I}$, find the values of x, y, z, and q.

[2 marks]

Question 1b

b)
Given that $\frac{1}{2}\boldsymbol{E}=r\boldsymbol{C}+s\boldsymbol{D}$ find the values of a, b, r, and s.

[3 marks]

Question 1c

c)
Given that $e\boldsymbol{M}-f\boldsymbol{N}=\boldsymbol{I}$ find the values of c, d, e, and f.

[3 marks]

Question 2

Consider the two matrices

$$\boldsymbol{A} = \begin{pmatrix} 3 & 2 \\ 2 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 13 & 6 \\ 6 & q \end{pmatrix}$$

where $q \in \mathbb{R}$ is a constant.

Given that $\boldsymbol{A}$ and $\boldsymbol{B}$ are commutative, find the value of q.

[3 marks]

Question 3

Consider the matrix

$$\boldsymbol{M} = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$$

Find an expression for $\boldsymbol{M}^k$.

[3 marks]

Question 4

Consider the matrices:

$$\boldsymbol{A}=\begin{pmatrix}2 & a\\ 3 & 4\end{pmatrix} \quad \boldsymbol{B}=\begin{pmatrix}1 & 3 & 4\\ 3 & -2 & b\end{pmatrix} \quad \boldsymbol{C}=\begin{pmatrix}1 & 2 & 3\\ 4 & c & 5\\ -5 & -3 & -2\end{pmatrix}$$

$$\boldsymbol{D}=\begin{pmatrix}d & 1\\ 2 & 3\end{pmatrix} \quad \boldsymbol{E}=\begin{pmatrix}3 & 2 & -1\\ e & -2 & 4\\ 1 & -3 & 2\end{pmatrix} \quad \boldsymbol{F}=\begin{pmatrix}2 & f\\ -2 & -1\\ 3 & 2\end{pmatrix}$$

where $a, b, c, d\ e, f \in \mathbb{R}$ are constants.

Find the following products in terms of the appropriate constants. If it is not possible to do so, explain why.

(i)
$\boldsymbol{AB}$
(ii)
$\boldsymbol{CA}$
(iii)
$\boldsymbol{EB}$
(iv)
$\boldsymbol{BE}$
(v)
$\boldsymbol{EA}+\boldsymbol{F}$
(vi)
$\boldsymbol{F}(\mathrm{C}-\mathrm{E})$
(vii)
$(\boldsymbol{C}-\boldsymbol{E})\boldsymbol{F}$

[8 marks]

Question 5

Consider the following matrices:

$$\boldsymbol{M}=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \boldsymbol{N}=\begin{pmatrix} e & f \\ g & h \end{pmatrix} \quad \boldsymbol{P}=\begin{pmatrix} i & j \\ k & l \end{pmatrix}$$

Show that $(\boldsymbol{MN})\boldsymbol{P}=\boldsymbol{M}(\boldsymbol{NP})$ and state the name of this property.

[5 marks]

Question 6a

For each of the following matrices,

(i) find the values of x for which $\boldsymbol{M}^{-1}$ does not exist, and

(ii) for the cases where $\boldsymbol{M}^{-1}$ does exist, find $\boldsymbol{M}^{-1}$ in terms of x.

a)

$$\boldsymbol{M}=\begin{pmatrix} x & -2 \\ 1 & 3 \end{pmatrix}$$

[2 marks]

Question 6b

b)

$$\boldsymbol{M}=\begin{pmatrix} x-2 & 3 \\ 4 & 2x \end{pmatrix}$$

[2 marks]

Question 6c

c)

$$\boldsymbol{M}=\begin{pmatrix} x^2 & x-1 \\ 3x & 5 \end{pmatrix}$$

[2 marks]

Question 7a

A message is encoded using a matrix. Letters in the message are represented by numbers as given in the table below.

A	B	C	D	E	F	G	H	I	J	K	L	M
1	2	3	4	5	6	7	8	9	10	11	12	13

N	O	P	Q	R	S	T	U	V	W	X	Y	Z
14	15	16	17	18	19	20	21	22	23	24	25	26

Messages are encoded by splitting the message into pairs of letters and writing in a $2 \times n$ matrix. For example, "encode" becomes $\begin{pmatrix} E & C & D \\ N & O & E \end{pmatrix}$ or $\begin{pmatrix} 5 & 3 & 4 \\ 14 & 15 & 5 \end{pmatrix}$. Then the message is multiplied, on the left, by an encryption matrix.

Here is a message which has been encoded using the encryption matrix $\begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix}$:

$$\begin{pmatrix} 41 & 96 & 33 & 53 & 96 & 46 \\ -9 & 52 & 3 & 71 & 66 & 8 \end{pmatrix}$$

a)
Decode the message.

[4 marks]

Question 7b

A spy (whose name cannot be given) notices that if the message has an odd number of letters, it will not completely fill a $2 \times n$ matrix. The spy suggests putting the message into a $3 \times n$ matrix if the number of letters in the message is a multiple of three.

b)
Assuming the same encryption matrix is used as was used for the above message, explain the problem with the spy's suggestion.

[1 mark]

Question 8a

Using the properties of matrices, explain why the following misconceptions are incorrect.

a)
"I know that $(a+b)(a-b) = a^2 - b^2$, therefore $(\boldsymbol{A}+\boldsymbol{B})(\boldsymbol{A}-\boldsymbol{B}) = \boldsymbol{A}^2 - \boldsymbol{B}^2$ must also be true if A and B are matrices."

[2 marks]

Question 8b

b)
"I know that $(a+b)^2 = a^2 + 2ab + b^2$, therefore $(\boldsymbol{A}+\boldsymbol{B})^2 = \boldsymbol{A}^2 + 2\boldsymbol{AB} + \boldsymbol{B}^2$ must also be true if A and B are matrices."

[2 marks]

Question 9a

In this question $\boldsymbol{A}$ and $\boldsymbol{B}$ are 2×2 matrices, $k, p, q \in \mathbb{R}$ are constants, and n is a positive integer.

a)
i)
Show that $(p\boldsymbol{A})(q\boldsymbol{B}) = pq(\boldsymbol{AB})$.
ii)
Hence show that $(k\boldsymbol{A})^n = k^n \boldsymbol{A}^n$.

[5 marks]

Question 9b

A student claims it is always true that $(\boldsymbol{AB})'' = \boldsymbol{A}''\boldsymbol{B}''$.

b)
Either explain why the student's claim is always true, or else show that it is not always true by providing a counterexample.

[2 marks]

Question 10a

Consider the following matrices:

$$\boldsymbol{A}=\begin{pmatrix}5 & p\\ p & 7\end{pmatrix} \quad \boldsymbol{B}=\begin{pmatrix}11 & q\\ q & 3\end{pmatrix}$$

where $p, q \in \mathbb{R}$ are constants, $q > 0$.

a)
Given that $\boldsymbol{A}$ and $\boldsymbol{B}$ are commutative, find q in terms of p.

[3 marks]

Question 10b

b)
Given that the determinant of $\boldsymbol{A}$ is 26, find p and q.

[2 marks]

Question 11a

A couple are planning their wedding reception and wish to buy a bow for every chair, with a matching tablecloth for each table.

There are a total of 120 chairs and 15 tables, all of which need bows and tablecloths respectively.

Company A charges £1.03 per chair bow and £14 per tablecloth.

Company B charges £0.85 per chair bow and £16 per tablecloth.

a)
By setting up one matrix equation that includes both companies, compare the overall prices that would be charged by the two companies.

[3 marks]

Question 11b

Married friends of the couple recommend Company C, whom they used for their wedding. The friends can remember that they paid £13 per tablecloth, but cannot remember the price per chair bow.

b)
Set up and solve a matrix equation to find the maximum price per chair bow that Company C could charge so as still to be cheaper overall than companies A and B.

[3 marks]

Question 12a

Veterinarians, veterinary nurses, and animal care assistants are paid fixed salaries according to an industry standard. The totals of the annual payrolls for three veterinary practices that pay according to the industry standard are summarised in the table below.

Practice	Veterinarians	Veterinary Nurses	Animal Care Assistants	Total Salary Spend
Aspen Road Vets	3	5	2	$294000
Broadoak Way Vets	2	2	1	$158000
Cats n Dogs Vets	7	10	4	$634000

a)
Using matrices, set up and solve a system of equations to find the fixed salaries that are paid for each of the three roles.

[4 marks]

Question 12b

Vicky is setting up a new veterinary practice, and to help recruit staff she is planning to pay 5% above the industry standard for all job roles. She uses the following matrix multiplication to help find the total cost of her staffing, where p, q and r represent the salaries for a veterinarian, a veterinary nurse, and an animal care assistant respectively:

$$a \times \begin{pmatrix} 3 & 4 & 2 \end{pmatrix} \times \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \text{Total salary spend in thousands}$$

b)

(i) Write down the value of the constant a that Vicky should use

(ii) Interpret the meaning of the element '4' in the row matrix.

[2 marks]

Question 13a

Consider a curve with equation $y = ax^3 + bx^2 + cx - 8$, where a, b, and c are real constants. The graph passes through the points $\mathrm{A}(2,\ 116)$, $\mathrm{B}(-4,\ -712)$ and $\mathrm{C}(3,\ 394)$.

a)

(i) Use a matrix method to find the values of a, b and c.

(ii) Hence sketch the curve.

[4 marks]

Question 13b

Consider a second curve with equation $ax^5 + bx^4 + cx^3 + dx^2 + ex + f$, where *a*, *b*, c, *d*, e and *f* are real constants with $a \neq 0$.

b)
By considering the method used to solve part (a), suggest the number of coordinates that would need to be known to determine the values of all the constants in the equation of the second curve.

[2 marks]