5.10 Differential Equations

Question Paper

Course	DP IB Maths
Section	5. Calculus
Topic	5.10 Differential Equations
Difficulty	Hard

Time allowed: 120

Score: /92

Percentage: /100

Question 1

Consider the first-order differential equation

$$\frac{dy}{dx} - x^3 = 2\sin x$$

Solve the equation given that $y = 0$ when $x = 0$, giving your answer in the form $y = f(x)$.

[5 marks]

Question 2a

Use separation of variables to solve each of the following differential equations:

(a)

$$\frac{dy}{dx} = 10x^3y^3$$

[4 marks]

Question 2b

(b)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x(x^2 - 1)^3 e^{3y}$$

[5 marks]

Question 3a

Use separation of variables to solve each of the following differential equations for y which satisfies the given boundary condition:

(a)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos 3x}{y}; \quad y\left(\frac{\pi}{6}\right) = -1$$

[5 marks]

Question 3b

(b)

$$e^{2x}\frac{dy}{dx} = \cos^2 y; \quad y(0) = \frac{\pi}{4}$$

[5 marks]

Question 4a

After an invasive species of insect has been introduced to a new region, it is estimated that at any point in time the rate of growth of the population of insects in the region will be proportional to the current population size P. At the start of a study of the insects in a particular region, researchers estimate the population size to be 1000 individuals. A week later another population survey is conducted, and the population of insects is found to have increased to 1150.

(a)

By first writing and solving an appropriate differential equation, determine how long it will take for the population of insects in the region to increase to 10 000.

[8 marks]

Question 4b

(b)
Comment on the validity of the model for large values of t.

[2 marks]

Question 5a

Ignoring the advice of her father's professional dragon keepers, Princess Sarff releases her personal menagerie of 800 dragons onto the archipelago known as the Sheep Islands. Sarff believes that the dragons will thrive in such a sheep-rich environment. The chief dragon keeper, however, has studied the sheep population of the islands as well as the appetite of dragons. Based on his research, he believes that the population P of dragons in the islands may be modelled by the logistic equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = 0.00025P(160 - P)$$

where t is the time in years after the dragons were introduced to the archipelago.

(a)
Use the logistic equation to explain why, according to the model, the dragon population will initially be decreasing.

[2 marks]

Question 5b

(b)
By first solving the logistic equation for P, determine the amount of time it will take for the dragon population to shrink to half its original size.

[10 marks]

Question 5c

(c)
Determine the long-term trend for the dragon population, using mathematical reasoning to justify your answer.

[3 marks]

Question 6a

Consider the differential equation

$$(x^2 + y^2)\frac{\mathrm{d}y}{\mathrm{d}x} = xy$$

(a)

Explain why the substitution $v = \dfrac{y}{x}$ would be an appropriate method to use to solve the differential equation.

[2 marks]

Question 6b

(b)
Show that the solution to the differential equation may be expressed in the form

$$y = Ae^{\frac{x^2}{2y^2}}$$

where A is an arbitrary constant.

[5 marks]

Question 6c

(c)

Find the precise solution to the differential equation given that $y = \frac{1}{2}$ when $x = 1$.

[3 marks]

Question 7

Use the substitution $v = \frac{y}{x}$ to solve the differential equation

$$x^2 y' = y^2 + 7xy + 9x^2$$

for y which satisfies the boundary condition $y(1) = -2$. Give your answer in the form $y = f(x)$.

[8 marks]

Question 8

Use an integrating factor to solve the differential equation

$$xy' + 2y = 1 + e^{x^2}$$

[6 marks]

Question 9a

Consider the differential equation

$$\frac{dy}{dx} = \left(\frac{\sec x}{e^{\sqrt{x}}}\right)^2 - \frac{y}{\sqrt{x}}$$

with the boundary condition $y\left(\frac{\pi}{3}\right) = 0$.

(a)

Apply Euler's method with a step size of $h = 0.01$ to approximate the solution to the differential equation at $x = \frac{20\pi + 3}{60}$.

[3 marks]

Question 9b

(b)

Solve the differential equation analytically, for y which satisfies the given boundary condition.

[7 marks]

Question 9c

(c)

(i)

Compare your approximation from part (a) to the exact value of the solution at $x = \frac{20\pi + 3}{60}$.

(ii)

Explain how the accuracy of the approximation in part (a) could be improved.

[3 marks]

Question 10a

A particle moves in a straight line, such that its displacement x at time t is described by the differential equation

$$\frac{dx}{dt} = \frac{1}{1 + \sin(t+1) - \cos(t+1)}, \qquad 0 \le t \le 3.5$$

At time $t = 2$, $x = 1$.

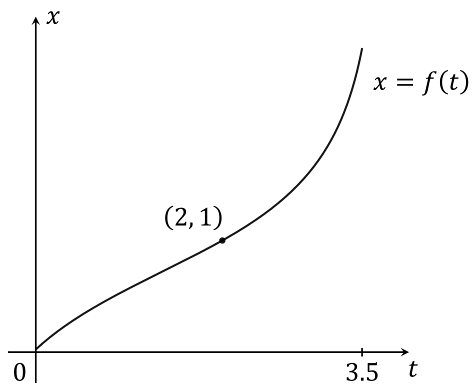
(a)

By using Euler's method with a step length of 0.25, find an approximate value for x at time $t = 3.25$.

[3 marks]

Question 10b

The diagram below shows a graph of the exact solution $x = f(t)$ to the differential equation with the given boundary condition.



(b)
Explain using the graph whether the approximation found in part (a) will be an overestimate or an underestimate for the true value of x when $t = 3.25$. Be sure to use mathematical reasoning to justify your answer.

[3 marks]