1.9 Further Complex Numbers

Question Paper

Course	DP IB Maths
Section	1. Number & Algebra
Topic	1.9 Further Complex Numbers
Difficulty	Very Hard

Time allowed: 100

Score: /82

Percentage: /100

Question 1a

Consider the equation $pz^3 + qz^2 + 8p^3z + 5q = 0$, where $z \in \mathbb{C},\ p \in \mathbb{R}$.

(a)

Given that one of the distinct roots is $z_1 = \frac{5}{2}$, find

(i)
the values of p and q
(ii)
the roots z_1 and z_2, giving your answers in the form $a + b\mathrm{i}$, where $a, b \in \mathbb{R}$.

[6 marks]

Question 1b

On an Argand diagram $z_1,\ z_2$ and z_3 are represented by the points A, B and C respectively.

(b)
Find the area of the triangle ABC.

[2 marks]

Question 2

Consider the complex numbers z and w, where $z = \sqrt{3} - \mathrm{i}$, $\mathrm{Im}\left(\frac{z^2}{w}\right) = 0$, $\left|\frac{z^2}{w}\right| = \frac{1}{2}|z|$.

Use geometrical reasoning to find the two possibilities for w, giving your answers in exponential form.

[4 marks]

Question 3

Consider the equation $z^4 - 5az^3 + 25az^2 - 20abz + 24ab = 0$, where $a,\ b \in \mathbb{Z}$ and $z \in \mathbb{C}$.

Given that one root is $a + a\mathrm{i}$ and another root is $b + b\mathrm{i}$, find the possible values of a and b.

[8 marks]

Question 4a

Consider the complex number $z = 1 + \sqrt{3}\,\mathrm{i}$.

(a)
Use De Moivre's theorem to find the value of z^3.

[3 marks]

Question 4b

(b)
Use the principle of mathematical induction to prove, for all $n \in \mathbb{N}$, that

$$(\cos\,\theta + \mathrm{i}\,\sin\,\theta)^n = \cos\,n\theta + \mathrm{i}\,\sin\,n\theta$$

[5 marks]

Question 4c

(c)
Show that the result in part (b) is true for all $n \in \mathbb{Z}$.

[2 marks]

Question 5a

Consider the equation $(z + a)^5 + 1 = 0,\ z \in \mathbb{C}$.

(a)
Given that the product of the roots is 31, find the roots of the equation, expressing your answers in the form $\omega_n = b + e^{i\theta}$, where $b \in \mathbb{R}$ and $\theta > 0$.

[5 marks]

Question 5b

Let S be the sum of the roots found in part (a).

(b)
Show that $\mathrm{Im}(S) = 0$ and find the value of $\mathrm{Re}(S)$.

[3 marks]

Question 5c

The roots $\omega_1,\ \omega_2, \dots,\ \omega_5$ are represented on an Argand diagram.

(c)
Describe the geometrical shape made by the five roots.

[1 mark]

Question 6

Consider the equations $u^* + 2v = 2\text{i}$ and $\text{i}u + v^* = 3$, where $u,\ v \in \mathbb{C}$. Find $\frac{u}{v}$ giving your answer in the form $r\text{e}^{\text{i}\theta}$, where $r > 0$ and $0 < \theta < 2\pi$.

[8 marks]

Question 7

By first expressing $1+\sqrt{3}\text{i}$ and $-1+\text{i}$ in the form $r\,\text{cis}\,\theta$ where $r>0$ and $-\pi<\theta\leq\pi$, show that $\frac{5\pi}{12}=2+\sqrt{3}$.

[8 marks]

Question 8a

Consider the complex number $z=1+\cos 2\theta+\text{i}\sin 2\theta$.

(a)
Find the modulus and argument of *z*.

[5 marks]

Question 8b

(b)
Solve $z = 0$ for $-\pi < \theta \leq \pi$.

[2 marks]

Question 9a

Let $z = e^{i\theta}$.

(a)
Show that
(i)
$z + z^2 + z^3 + \cdots + z^n = \cos\theta + \cos 2\theta + \cdots + \cos n\theta + i(\sin\theta + \sin 2\theta + \cdots + \sin n\theta)$,
(ii)
$(2 - z)(2 - z^*) = 5 - 4\cos\theta$.

[4 marks]

Question 9b

(b)
Use the results found in part (a) to find the sum of the infinite series

$$\frac{\sin\theta}{2}+\frac{\sin 2\theta}{2^2}+\frac{\sin 3\theta}{2^3}+\frac{\sin 4\theta}{2^4}+\ldots$$

[4 marks]

Question 10a

The primary square root of a complex number z is defined as $\sqrt{z}=x+\mathrm{i}y$, where $x,\ y\in\mathbb{R}$ and $x\geq 0$. If $x=0$ then the value for y is chosen such that $y\geq 0$. Note that the other square root of z will then be given by $-\sqrt{z}=-x-\mathrm{i}y$.

(a)
Show that

$$x=\sqrt{\frac{\mathrm{Re}(z)+\sqrt{(\mathrm{Re}(z))^2+(\mathrm{Im}(z))^2}}{2}}$$

[5 marks]

Question 10b

(b)
Given that $x > 0$, derive a formula for y in terms of x and $\mathrm{Im}(z)$, and explain why y in this case will always have the same sign (positive, negative, or zero) as $\mathrm{Im}(z)$.

[2 marks]

Question 10c

(c)
Hence show that in general

$$y = \pm\sqrt{\frac{-\mathrm{Re}(z) + \sqrt{(\mathrm{Re}(z))^2 + (\mathrm{Im}(z))^2}}{2}}$$

with the choice of the positive or negative value being dependent on the properties of z.

[2 marks]

Question 10d

(d)

Explain what must be true of z for each of the following to be true:

(i)

$x = 0,\ y \neq 0$

(ii)

$x \neq 0,\ y = 0$

(iii)

$x = 0,\ y = 0$

[3 marks]