4.13 Transition Matrices & Markov Chains

Question Paper

Course	DP IB Maths
Section	4. Statistics & Probability
Topic	4.13 Transition Matrices & Markov Chains
Difficulty	Hard

Time allowed: 90

Score: /74

Percentage: /100

Question 1a

It is known that in the legendary frozen town of Kholod the weather displays the following patterns:

- If it snows on one day then there is a probability of 0.8 that it will snow on the following day
- If it does not snow on one day then there is a probability of 0.6 that it will also not snow on the following day

(a) Represent this information as

(i) a transition state diagram

(ii) a transition matrix.

[4 marks]

Question 1b

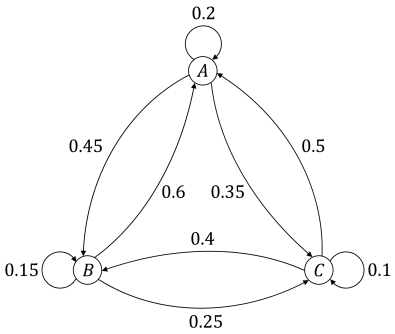
Let T be the transition matrix found in part (a)(ii).

b)
By first finding an appropriate power of the matrix T, find the probability that it will snow on Friday, given that it did not snow on the preceding Tuesday.

[4 marks]

Question 2a

Bartolomé is a professional probabilist who has devised a way to save money on lodging. He has decided he can spend each day working in cafés in one of three cities *A*,*B*, and *C*. Then each night he will sleep on an overnight train travelling between the cities (or in some cases returning by morning to the same city from which it left). Due to his fondness for probability, Bartolomé has determined to allow his movement between the cities each night to be subject to the probabilities in the following transition state diagram:



a)
Write down the transition matrix T for this system of probabilities.

[2 marks]

Question 2b

b)
One Friday, Bartolomé spends the day in the same city where he spent the preceding Monday. Determine the probabilities of this having happened for each of the cities.

[4 marks]

Question 2c

c)
Find the long-term probabilities of Bartolomé spending a given day in cities A, B or C.

[3 marks]

Question 3a

Two bands from Wem, WeDaBoyz and Grrl Pow-R, are the megastars of the new W-Pop scene. At any one time, no fan of WeDaBoyz will ever also be a fan of Grrl Pow-R, and no fan of Grrl Pow-R will ever also be a fan of WeDaBoyz. Each week, however, 17% of WeDaBoyz's fans switch to become fans of Grrl Pow-R, and 13% of Grrl Pow-R's fans switch to become fans of WeDaBoyz. It may be assumed that there are no other gains or losses of fans by the two bands.

a)
Write down a transition matrix T representing the movement of fans between the two bands in a particular week.

[2 marks]

Question 3b

Initially WeDaBoyz has 4000 fans and Grrl Pow-R has 2300 fans.

b)
Write down the initial state vector s_0 for the system.

[1 mark]

Question 3c

c)
Determine the number of fans that WeDaBoyz and Grrl Pow-R will each have after four weeks.

[3 marks]

Question 3d

d)
Find
(i)
the number of followers that WeDaBoyz and Grrl Pow-R will each have in the long term,
(ii)
the total number of people per week in the long term who will change from being a fan of one band to being a fan of the other.

[5 marks]

Question 4a

On a mystery cruise, each passenger is assigned to one of three groups – suspects, investigators or red herrings. At a special 'reveal' event each night, passengers are reassigned roles according to the following scheme:

- Due to additional revealed information about the fictional crime, half the suspects become red herrings while one twentieth of them become investigators
- Due to revealed events from their fictional pasts, one third of the investigators become suspects and one tenth become red herrings
- Due to revealed disambiguations, all of the red herrings become either suspects or investigators, with five becoming suspects for each two that become investigators

There is no other way for the numbers of passengers in each of the three roles to increase or decrease. Initially there are 2100 suspects, 15 investigators, and 885 red herrings.

a)
Write down a transition matrix T and an initial state vector s_0 representing the movement of passengers between the three roles from night to night.

[4 marks]

Question 4b

b)
Determine the number of nightly reveals will it take until the number of investigators exceeds 650.

[3 marks]

Question 4c

c)
If the cruise continued for an indefinite period of time, find
(i)
the number of passengers in the long term who would end up assigned to each role at the end of a given night,
(ii)
the number of passengers who would not change roles on any given night.

[5 marks]

Question 5a

The marketing department of BreadHead brand shampoo ("It makes you look like you have breadcrumbs in your hair!") is attempting to predict the percentage of potential customers who will purchase its product month by month in the future.

The lead marketing researcher reports that, according to his research, there is only a 3% chance that a potential customer who buys BreadHead shampoo one month will buy it again the following month. On the other hand, due to product placement deals with a number of popular social media influencers, there is a 99% chance that a potential customer who does not buy BreadHead shampoo one month will buy it the following month.

a)
Explain what assumption the marketing researcher would need to make in order to use a Markov chain to predict month by month sales according to his research.

[1 mark]

Question 5b

The researcher decides to model the situation using a Markov chain. It is known that currently 5% of potential customers buy BreadHead shampoo.

b)
Find the probability that a randomly selected potential customer

(i) will purchase BreadHead shampoo next month,

(ii) will purchase BreadHead shampoo in the long term.

[5 marks]

Question 5c

c)
Criticise the model, in particular the researcher's decision to use a Markov chain.

[3 marks]

Question 6a

An electric car hire company offers one-day hires from three hire centres located in a large city – one at the Airport (A), one at the Breakdancing Museum (B), and one in the City Centre (C). Past experience shows that 82% of cars hired at A are returned to A; otherwise they are twice as likely to be returned to C as they are to B. Only 12% of cars hired at B are returned to B, with 48% being returned to A and the rest returned to C. Of cars hired at C, 64% are returned to C, with equal numbers of the remainder returned to A and B.

a)
By solving a system of linear equations represented by $Tp = p$, for an appropriately defined matrix T, determine a steady state vector p corresponding to T.

[5 marks]

Question 6b

After a one-week shutdown for maintenance, the company is about to redistribute its entire fleet of 725 cars between its three locations.

b)
(i)
Suggest how the answer to part (a) might be used by the company to decide how many cars should initially be placed in locations A,B and C. Be sure to justify your answer.
(ii)
Explain why the distribution of cars in part (b)(i) might not necessarily be the best way for the company to distribute their hire cars.

[4 marks]

Question 7a

In the town of Petersham, all the residents belong to either one or the other of the town's two curling clubs – the Slippery Sliders (S) or the Wild Sweepers (W). Competition between the clubs is fierce, and members tend to be quite loyal. Still, each year 2% of the members of S switch to W and 3% of the members of W switch to S. Any other losses or gains of members by the two curling clubs may be ignored.

a)
Write down a transition matrix $\boldsymbol{T}$ representing the movement of members between the two clubs in a particular year.

[2 marks]

Question 7b

b)
Diagonalise the matrix $\boldsymbol{T}$ by writing it in the form $\boldsymbol{T = PDP^{-1}}$ for appropriate matrices $\boldsymbol{P}$ and $\boldsymbol{D}$.

[6 marks]

Question 7c

Initially there are 574 members of and 621 members of W.

c)
Show that the numbers of members of S and W after n years will be $(717 - 143(0.95^n))$ and $(478 + 143(0.95^n))$, respectively.

[6 marks]

Question 7d

d)
Hence write down the number of members that each of the curling clubs can expect to have in the long term.

[2 marks]