1.7 Matrices

Question Paper

Course	DP IB Maths
Section	1. Number & Algebra
Topic	1.7 Matrices
Difficulty	Medium

Time allowed: 130

Score: /105

Percentage: /100

Question 1a

Consider the following matrices:

$$A = \begin{pmatrix} 1 & 0 & -2 \\ -1 & a & 3 \end{pmatrix} \qquad B = \begin{pmatrix} b & 2 \\ 1 & -1 \end{pmatrix} \qquad C = \begin{pmatrix} 0 & c & -3 \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 1 & 0 \\ 3 & d & 1 \\ 0 & 3 & 0 \end{pmatrix} \qquad E = \begin{pmatrix} -1 & 4 & e \\ -2 & 1 & 0 \end{pmatrix} \qquad F = \begin{pmatrix} f \\ 2 \\ -1 \end{pmatrix}$$

where $a, b, c, d, e, f \in \mathbb{R}$ are constants.

(a)
Find each of the following matrix sums or differences, or if that is not possible then explain why:

(i) $A + B$ (ii) $A + E$ (iii) $E - 3A$

[4 marks]

Question 1b

(b)
Find each of the following matrix products, or if that is not possible then explain why:

(i) BE (ii) EB (iii) CD (iv) FC (v) AE

[5 marks]

Question 1c

(c)

List any other matrix products of two of the above matrices that it *would* be possible to find, other than the ones included in part (b). You do not need to find the products, but do specify what the order of each of those products would be.

[3 marks]

Question 1d

(d)

Find each of the following:

(i) $B(A + E)$ (ii) $BA + BE$

[3 marks]

Question 2a

Consider the matrices:

$$A = \begin{pmatrix} 2 & -1 \\ 4 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 \\ -4 & k \end{pmatrix}$$

where $k \in \mathbb{R}$ is a constant.

Let I be the 2×2 identity matrix, and let 0 be the 2×2 zero matrix.

(a)
Find the following

(i) B^2 (ii) $A - 2I$ (iii) AB (iv) BA (v) $A^{-1}B$

[7 marks]

Question 2b

(b)
Find the following:

(i) $\det A$ (ii) $\det A^{-1}$ (iii) $\det B$ (iv) $\det AB$

[4 marks]

Question 2c

C and D are matrices such that $A + C = 0$ and $B^{-1} + D = 0$

(c)

(i)   Write down matrix C.

(ii)
Find matrix D.

[4 marks]

Question 3a

Consider the matrix

$$A = \begin{pmatrix} 3 & -8 \\ p & 7 \end{pmatrix}$$

where $p \in \mathbb{R}$ is a constant.

(a)
Given that $\det A = -3$, find the value of p.

[3 marks]

Question 3b

Consider the two matrices

$$B = \begin{pmatrix} 4 & q \\ -1 & 3 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} r & 1 \\ 1 & 3 \end{pmatrix}$$

where $q, r \in \mathbb{R}$ are constants.

(b)
Given that $BC = CB$, find the values of q and r.

[4 marks]

Question 4a

Consider the matrices $P = \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 \\ 0 & 0.8 \end{pmatrix}$.

(a)
Find

(i)   the determinant of P

(ii) P^{-1}.

[2 marks]

Question 4b

Consider the matrix product PDP^{-1}.

(b)
(i)
Show that $(PDP^{-1})^2 = PD^2P^{-1}$ and $(PDP^{-1})^3 = PD^3P^{-1}$.

(ii)
Use the results of part (b)(i) to suggest an expression for $(PDP^{-1})^n$ in terms of P, D and P^{-1}.

[3 marks]

Question 4c

The transition matrix of a dynamic system is $T = \begin{pmatrix} 0.88 & 0.08 \\ 0.12 & 0.92 \end{pmatrix}$.

c)
Find T^5.

[1 mark]

Question 4d

(d)

(i)

Show that $T = PDP^{-1}$.

(ii)

Hence use the answer to part (c) to confirm the validity of your expression from part (b)(ii) for $n = 5$.

[4 marks]

Question 5a

A family are buying burgers for dinner and wish to order three veggie burgers, one beef burger and two chicken burgers.

From burger store A three veggie burgers would cost a total of $15.45, one beef burger would cost $6.15, and 2 chicken burgers would cost a total of $11.90.

(a)

Write down

(i)

a row matrix, Q, to represent the quantities of each type of burger that the family wishes to purchase

(ii)

a column matrix, P_A, to represent the cost of each type of burger at store A.

[1 mark]

Question 5b

Burger store B sells their veggie burger, beef burger and chicken burger for $4.75, $5.85 and $5.50, respectively.

(b)

(i)

Write down a column matrix, P_B, to represent the cost of each type of burger at store B.

(ii)

Hence write down a cost matrix, C, to represent the costs for both stores.

[3 marks]

Question 5c

(c)

By first calculating the matrix QC, compare the total cost of the family's dinner at stores A and B.

[2 marks]

Question 6a

A café is looking to hire a duty manager, two baristas, a dishwasher, and three waiters. They decide to advertise the jobs on social media, as well as using a hiring agency. They receive 123 applications from advertising the job on social media and 57 applications from the hiring agency.

(a)

Write down a column matrix, C, to represent the number of applications they received from advertising the job on social media and from using the hiring agency.

[1 mark]

Question 6b

Overall, 22% of the applicants applied for the duty manager job, 28% applied for the barista job, 18% applied for the dishwasher job, and the rest applied for the waiter job. Note that every applicant was only allowed to apply for one of the available jobs.

(b)
Write down a row matrix, R, to represent the percentages of the applicants that applied for each of the different jobs.

[1 mark]

Question 6c

(c)
(i)
Calculate the product $P = CR$.

(ii)
Use the elements of the matrix P to work out the total number of applicants for each of the positions, giving your answers to the nearest integer.

[3 marks]

Question 6d

Once the café has selected the right candidate for each job, each new employee will work a total of 40 hours per week. The hourly wage for the duty manager and barista jobs is $20.00 per hour, and the hourly wage for the dishwasher and waiter jobs is $17.25.

(d)
Calculate the café's weekly wage expenses for the new employees.

[2 marks]

Question 7a

Amanda is a landlord who has new sets of tenants moving into three new unfurnished homes. Amanda receives a special deal at a small local furniture store and so she offers her tenants that she will buy some tables, chairs, and/or sofas on their behalf, given that they reimburse her for the cost. The quantities of different items ordered for each house are shown in the table below.

	Tables	Chairs	Sofas
House 1	2	5	2
House 2	1	3	0
House 3	1	4	1

(a)

(i)

Write down a 3×3 matrix, F, to represent the furniture orders for the three houses.

(ii)

Write down a 1×3 row matrix, S, to represent Amanda's total shopping list at the furniture store.

[2 marks]

Question 7b

The price Amanda pays for each item is shown in the table below.

	Tables	Chairs	Sofas
Price	$72.00	$14.50	$47.50

(b)

(i)

By performing an appropriate matrix multiplication with matrix F, find the total amount owed to Amanda by each house.

(ii)

By performing an appropriate matrix multiplication with matrix S, find the total amount paid by Amanda to the furniture store.

[5 marks]

Question 8a

The bus fare a person pays in a city is dependent on whether the person is a student, an adult or a pensioner.

The total amount taken in on a particular day by three different buses, along with the numbers of each type of fare paid, are shown in the table below.

	Student	Adult	Pensioner	Total
Bus A	91	82	13	$348
Bus B	102	80	4	$355
Bus C	71	54	11	$247

Let s, a and p represent the amount paid by a student, an adult, and a pensioner respectively.

(a)
Write down a system of three linear equations in terms of s, a and p that represent the information shown in the table above.

[2 marks]

Question 8b

(b)
Find the values of s, a and p using appropriate matrices and matrix inverses.

[2 marks]

Question 8c

Bus D finished the day having sold 112 student tickets, 91 adult tickets and 22 pensioner tickets.

(c)
Calculate the total amount taken in by Bus D. Give your answer to 2 decimal places.

[2 marks]

Question 9a

The number of times a gamer logged in to Call of Duty, FIFA, or Assassin's Creed over 3 weeks is shown in the table below.

	Call of Duty	FIFA	Assassin's Creed
Week 1	5	4	3
Week 2	7	2	4
Week 3	3	5	6

The total number of hours the gamer spent playing each week is shown in the table below.

	Week 1	Week 2	Week 3
Total hours	12.35	13.84	14.16

The gamer was never logged in to more than one game at the same time.

The gamer believes that, for each game, the average amount of time spent playing per log-in session was consistent over the three weeks.

(a)
Assuming that the gamer's belief is true, use matrix multiplication to find the average number of hours and minutes per log-in session that the gamer spent playing each game.

[3 marks]

Question 9b

(b)
Write down a system of linear equations that could be used to find the answers in part (a).

[2 marks]

Question 10a

The graph of the quadratic function $f(x) = ax^2 + bx + c$ passes through the points $(-1, 11)$, $(3, -5)$ and $(6, 4)$.

(a) Show that a, b and c must satisfy the following system of linear equations:

$$a - b + c = 11$$

$$9a + 3b + c = -5$$

$$36a + 6b + c = 4$$

[2 marks]

Question 10b

(b)
Represent the system of equations in part (a) in matrix form.

[1 mark]

Question 10c

(c)
Hence use a matrix method to find the values of a, b and c.

[3 marks]

Question 11a

The graph of the function $f(x) = ax^2 + bx + c$ passes through the points $(-1, 5)$, $(3,1)$ and $(4, -5)$.

(a)
Write down a system of linear equations that a, b and c must satisfy.

[2 marks]

Question 11b

(b)
Hence use a matrix method to determine the values of a, b and c.

[4 marks]

Question 12a

The amounts of wheat, soybeans and sugar produced by three different farms in a given week, along with the respective total revenues for each farm, are shown in the table below.

	Wheat, kg	Soybeans, kg	Sugar, kg	Revenue, $
Farm A	820	532	535	835.54
Farm B	1210	641	274	948.75
Farm C	922	211	503	716.11

Let x, y and z represent the prices, in \$/kg, for wheat, soybeans and sugar respectively.

(a)
(i)
Write down a system of linear equations that represents the information in the table above.

(ii)
Solve the system of linear equations using matrices.

[4 marks]

Question 12b

In the same week, Farm D produced a fifth of the amount of wheat as Farm A, a quarter of the amount of soybeans as Farm B, and half the amount of sugar as Farm C.

(b)
Calculate the revenue made by Farm D from selling these crops. Give your answer correct to two decimal places.

[3 marks]

Question 13a

Grace has decided that she wants to invest split between three companies: company A, company B, and company C. She creates three different portfolio options based on risk levels, and calculates what each option's value would be today if the identical amounts had been invested one year ago.

	Company A	Company B	Company C	Value
Safe	$1500	$8000	$500	$10,620.00
Middle	$2000	$6750	$1250	$10,827.50
Risky	$2500	$2500	$5000	$11,725.00

(a)
Use a matrix method to find the annual percentage return (i.e., the percentage increase or decrease of an investment in the company) for the previous year for each company.

[6 marks]

Question 13b

Grace hears some good news about the growth of company A before she invests her money, and so she decides to put 78% of it into company A and split the rest evenly between company B and company C.

Compared with the previous year, the annual percentage return for company A for the coming year is expected to increase by 26 percentage points (so if the previous year's return was x%, then the return is expected to be $(x + 26)$% for the coming year). For company B the return is expected to remain the same, while for company C it is expected to decrease by 8 percentage points.

(b)
Find the expected value of Grace's investment at the end of the coming year.

[2 marks]