3.10 Vector Equations of Lines

Question Paper

Course	DP IB Maths
Section	3. Geometry & Trigonometry
Topic	3.10 Vector Equations of Lines
Difficulty	Hard

Time allowed: 100

Score: /80

Percentage: /100

Question 1a

Point A has coordinates $(7, -1, 20)$ and the line l is defined by the equations:

$$l: \begin{cases} x = 3 + \lambda \\ y = 2\lambda - 1 \\ z = \lambda \end{cases}$$

Point B lies on the line l such that $[\mathrm{AB}]$ is perpendicular to l.

(a)
Find the coordinates of point B.

[5 marks]

Question 1b

(b)
Hence find the shortest distance from A to the line l.

[2 marks]

Question 2a

(a)
Find the vector equation of the line l_1 with Cartesian equations $\frac{x+3}{2} = \frac{y-2}{5} = \frac{3-z}{-4}$

[2 marks]

Question 2b

A second line l_2 runs parallel to l_1 and passes through the points $\mathrm{X}(t, 2, -3)$ and $\mathrm{Y}(23, 22, q)$.

(b)
Find the value of t and q

[4 marks]

Question 2c

(c)
Hence write down the equation of line l_2 in Cartesian form.

[2 marks]

Question 3

A line l passes through the points $\mathrm{P}(6, 5, -2)$ and $Q(2x + 2, x - 5, x)$ and lies normal to the vector $3\mathrm{i} + 4\mathrm{j} - k$.

Find the vector equation of l.

[6 marks]

Question 4

Find the obtuse angle formed by the two lines l_1 and l_2 defined by the equations:

$$l_1: \begin{cases} x = 4 - 2\lambda \\ y = 1 + 5\lambda \\ z = \lambda - 1 \end{cases}$$

$$l_2: \begin{cases} x = 4 + 3\mu \\ y = 18 + \mu \\ z = 6 + 2\mu \end{cases}$$

[6 marks]

Question 5a

Consider the skew lines l_1 and l_2 as defined by:

$$l_1: \begin{cases} x = 5 + \mu \\ y = 3 - \mu \\ z = 2\mu - 8 \end{cases}$$

$$l_2: r = \begin{pmatrix} -4 \\ 3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -5 \\ 2 \end{pmatrix}$$

(a)
Find a vector that is perpendicular to both lines.

[3 marks]

Question 5b

(b)
Hence find the shortest distance between the two lines.

[5 marks]

Question 6

Consider the lines l_1 and l_2 defined by the equations:

$$l_1: \begin{cases} x = 2 + 6\lambda \\ y = 2 + q\lambda \\ z = -8 - 5\lambda \end{cases}$$

$$l_2: \frac{-4 - x}{24} = \frac{y - 5}{12} = \frac{z - p}{20}$$

Given that l_1 and l_2 are coincident, find the value of p and q.

[6 marks]

Question 7a

Two spaceships A and B, in a 3D virtual reality game, are moving such that their positions relative to a fixed point O at time t seconds, $0 \leq t < 30$, are defined by the position vectors $r_A = \begin{pmatrix} 2 \\ -3.5 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1.2 \\ 0.5 \\ 2 \end{pmatrix}$ and $r_B = \begin{pmatrix} -2 \\ 4 \\ 9.5 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \\ 0.3 \end{pmatrix}$ respectively.

(a)
Show that the two spaceships are on course to collide at point P and write down the coordinates of P.

[4 marks]

Question 7b

Spaceship B reduces its velocity such that its position vector is now given by

$$r_B = \begin{pmatrix} -2 \\ 4 \\ 9.5 \end{pmatrix} + t\begin{pmatrix} 1.6 \\ -0.8 \\ 0.24 \end{pmatrix}$$

(b)
Show that spaceship B is still travelling in its original direction.

[1 mark]

Question 7c

(c)
Show that the distance between the two spaceships can be written as

$$\sqrt{4.9476t^2 - 56.62t + 144.5}$$

[5 marks]

Question 7d

(d)
Hence find the distance between the two spaceships when spaceship A is at P.

[2 marks]

Question 8a

A car is moving with constant velocity along the line with equation $r_c = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + t\begin{pmatrix} 5 \\ 12 \end{pmatrix}$. A bird is perched at the point $(25,\ 32,8)$ and at $t = 0$, starts to fly at a constant velocity in the direction of the vector $(2\mathrm{i} + 31\mathrm{j} - 4k)$.

All distances are measured in metres and time in seconds. The base vectors i and j represent due east and due west respectively and the base vector $\boldsymbol{k}$ points upwards.

(a)
Verify that the bird does not collide with the car.

[2 marks]

Question 8b

(b)
Show that at some point in time the bird will be directly above the car and state the time at which this occurs.

[3 marks]

Question 8c

(c)
Hence find the distance between the bird and the car at that time.

[2 marks]

Question 9a

Consider the triangle ABC. The points A, B and C have coordinates $(-6, 3, 13)$, $(4, 5, -8)$ and $(3, -4, t)$ respectively. A vector equation of the line that passes through point A and the midpoint of $[\mathrm{BC}]$ is $r = \begin{pmatrix} -6 \\ 3 \\ 13 \end{pmatrix} + \lambda \begin{pmatrix} 19 \\ -5 \\ -27 \end{pmatrix}$

(a)
Find the value of t.

[3 marks]

Question 9b

(b)

Find the vector equation of the line that passes through point B and the midpoint of [AC].

[3 marks]

Question 9c

The two lines intersect inside the triangle at point X.

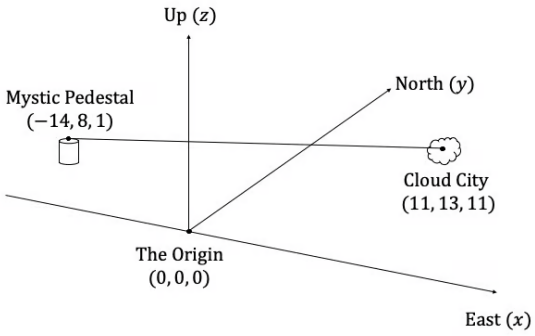
(c)

Show that the area of AXC is $\frac{1}{3}$ the area of triangle ABC.

[7 marks]

Question 10

In the magical kingdom of Cartesia, all positions are measured relative to the ancient stone of power known as the Origin. This reference system corresponds to the standard x, y, z coordinate system used in mathematics, as shown in the diagram below.



Prince Vector, son of the King Prime of Cartesia, needs to fly on his magical unicorn from the top of the Mystic Pedestal all the way to Cloud City, on an urgent rescue mission.

The Mystic Pedestal is 14 kilometres west and 8 kilometres north of the Origin, and its top is one kilometre up from the level of the Origin. Cloud City is 11 kilometres east and 13 kilometres north of the Origin, and it is 11 kilometres up from the level of the Origin.

Since there is not much time, the prince must fly directly from the top of the Mystic Pedestal to Cloud City. Unfortunately, the unicorn's magic levels are low. In order for the unicorn to recharge it must pass within 12 kilometres of the Origin during the flight, and must do this before reaching the halfway point between the Mystic Pedestal and Cloud City. If the unicorn does not recharge before this point then it and the prince will crash into the barren wastes and the kingdom will perish.

Using a vector method, determine whether or not the prince will reach Cloud City successfully. Use clear mathematical workings to justify your answer.

[7 marks]