5.5 Optimisation

Question Paper

Course	DP IB Maths
Section	5. Calculus
Topic	5.5 Optimisation
Difficulty	Medium

Time allowed: 110

Score: /84

Percentage: /100

Question 1a

A company manufactures food tins in the shape of cylinders which must have a constant volume of 150π cm^3. To lessen material costs the company would like to minimise the surface area of the tins.

(a) By first expressing the height h of the tin in terms of its radius r, show that the surface area of the cylinder is given by $S = 2\pi r^2 + \dfrac{300\pi}{r}$.

[2 marks]

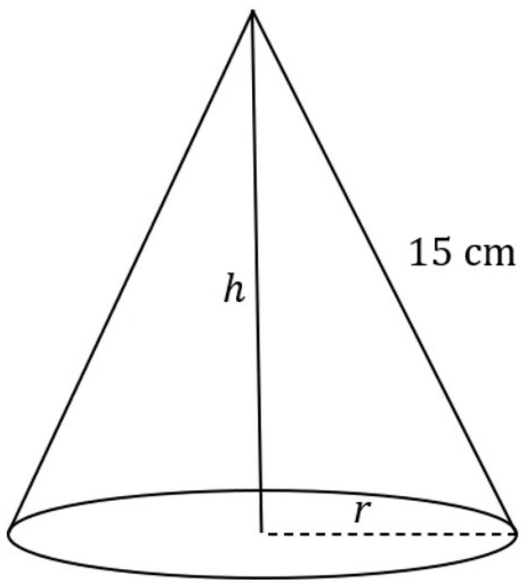
Question 1b

(b) Use calculus to find the minimum value for the surface area of the tins. Give your answer correct to 2 decimal places.

[4 marks]

Question 2a

A right-angled triangle of height h, base r and hypotenuse 15 cm has been rotated about its vertical axis to form a cone.



(a) Write an expression for r in terms of h.

[2 marks]

Question 2b

(b) Show that the volume of the cone, V cm^3, can be expressed as:

$$V = \frac{\pi}{3}(225h - h^3)$$

[3 marks]

Question 2c

(c) Find the value of h which provides the cone with its maximum volume.

[3 marks]

Question 3a

A wire of length 1 m is cut into two pieces. The first piece is bent into the shape of a square. The second piece is bent into a rectangle which has a length l twice as long as its width w. Let x cm be the perimeter of the square.

(a) Find an expression for the area of the square.

[3 marks]

Question 3b

(b) Show that the width of the rectangle $w = \frac{100-x}{6}$.

[3 marks]

Question 3c

(c) Find an expression for the sum of the area of the two shapes, S.

[3 marks]

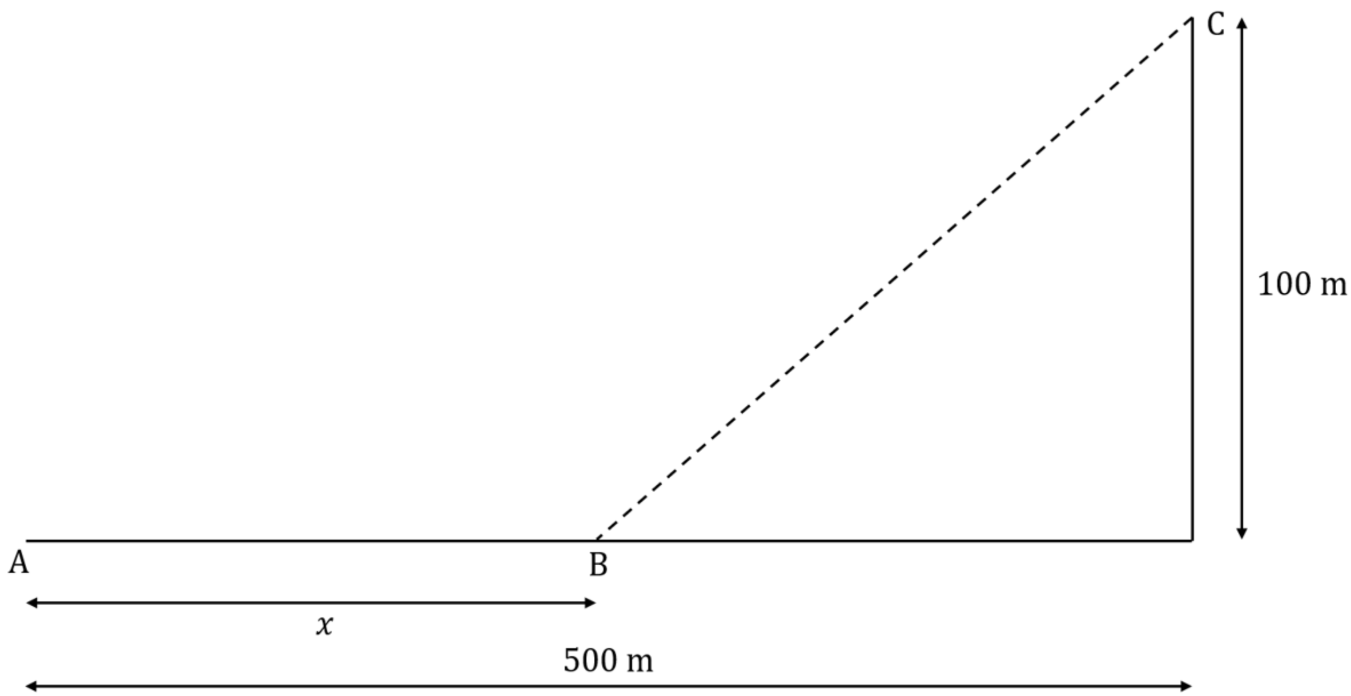
Question 3d

(d) Find the value of x such that the sum of the areas, S, is a minimum.

[4 marks]

Question 4a

Liam, a keen runner and swimmer, enters a competition which requires the competitors to run from point A along a straight beach, before diving into the water and swimming directly to point C. Liam is able to run at a speed of 8 m/s along the beach and swim at 2 m/s in the water.



Let x represent the distance between A and B, the distance that Liam runs along the beach before entering the water and swimming along the line BC.

(a) Find an expression for the time taken for Liam to run x metres between A and B.

[2 marks]

Question 4b

(b) Show that the length of $BC = \sqrt{10000 + (500 - x)^2}$.

[2 marks]

Question 4c

(c) Find an expression for the total time taken for Liam to finish the race.

[2 marks]

Question 4d

(d) Find the value of x that will allow Liam to complete the race in the quickest time.

[3 marks]

Question 5a

A small cylindrical drum, closed at the top but open at the bottom, has a radius r cm and a height h cm. The volume of the drum is 1000 cm^3.

The material to make the top skin of the drum costs 25 cents per cm^2 and the curved surface of the drum costs 20 cents per cm^2.

(a) Find an expression for h in terms of r.

[2 marks]

Question 5b

(b) Show that the total cost of the material to make the drum is $C = 25\pi r^2 + \frac{40000}{r}$.

[4 marks]

Question 5c

(c) Find $\frac{dC}{dr}$.

[2 marks]

Question 5d

The function $C(r)$ has a local minimum at the point (p, q).

(d) Find the value of p.

[4 marks]

Question 5e

(e) State, to the nearest dollar, the minimum cost required to make the drum.

[1 mark]

Question 5f

(f) Find $\frac{d^2C}{dr^2}$ and hence, describe the concavity of the function $C(r)$ at $x = p$.

[4 marks]

Question 6a

The daily cost function of a company producing pairs of running shoes is modelled by the cubic function

$$C(x) = 1225 + 11x - 0.009x^2 - 0.0001x^3, \qquad 0 \leq x < 160$$

where x is the number of pairs of running shoes produced and C the cost in USD.

(a) Write down the daily cost to the company if no pairs of running shoes are produced.

[1 mark]

Question 6b

The marginal cost of production is the cost of producing one additional unit. This can be approximated by the gradient of the cost function.

(b) Find an expression for the marginal cost, $C'(x)$, of producing pairs of running shoes.

[2 marks]

Question 6c

(c) Find the marginal cost of producing

(i) 40 pairs of running shoes

(ii) 90 pairs of running shoes.

[2 marks]

Question 6d

The optimum level of production is when marginal revenue, $R'(x)$, equals marginal cost, $C'(x)$. The marginal revenue, $R'(x)$, is equal to 4.5.

(d) Find the optimum level of production.

[3 marks]

Question 7a

A cyclist riding over a hill 3can be modelled by the function

$$h(t) = -\frac{1}{24}t^2 + 3t + 12, \qquad 0 \le t \le 70$$

where h is the cyclist's altitude above mean sea level, in metres, and t is the elapsed time, in seconds.

(a) Calculate the cyclist's altitude after a minute.

[2 marks]

Question 7b

(b) Find $h'(t)$.

[2 marks]

Question 7c

(c) Calculate the cyclist's maximum altitude and the time it takes to reach this altitude.

[3 marks]

Question 8a

A company produces and sells cricket bats. The company's daily cost, C, in Australian dollars (AUD), changes based on the number of cricket bats they produce per day. The daily cost function of the company can be modelled by

$$C(x) = 6x^3 - 10x^2 + 10x + 4$$

where x hundred cricket bats is the number of cricket bats produced on a particular day.

(a) Find the cost to the company for any day zero cricket bats are produced.

[1 mark]

Question 8b

The company's daily revenue, of AUD, from selling x hundred cricket bats is given by the function $R(x) = 42x$.

(b) Given that profit = revenue – cost, determine a function for the profit, $P(x)$, in hundreds of AUD from selling x hundred cricket bats.

[2 marks]

Question 8c

(c) Find $P'(x)$.

[2 marks]

Question 8d

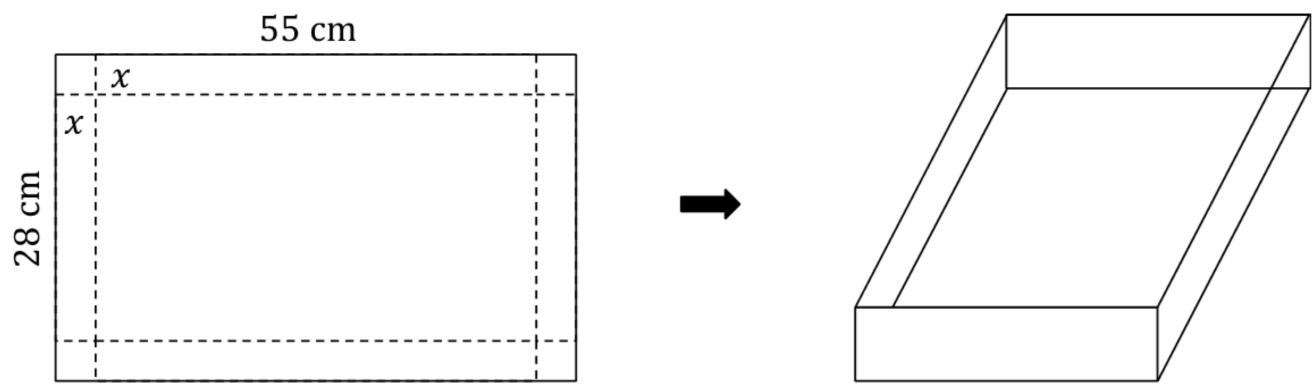
The derivative of $P(x)$ gives the marginal profit. The production of bats will reach its profit maximising level when the marginal profit equals zero and $P(x)$ is positive.

(d) Find the profit maximising production level and the expected profit.

[3 marks]

Question 9a

Dora decides to build a cardboard container for when she goes strawberry picking from a rectangular piece of cardboard, 55 cm $\times$ 28 cm. She cuts squares with side length x cm from each corner as shown in the diagram below.



(a) Show that the volume, V cm^3, of the container is given by

$$V = 4x^3 - 166x^2 + 1540x$$

[2 marks]

Question 9b

(b) Find $\frac{dV}{dx}$.

[2 marks]

Question 9c

(c) Find

(i) the value of x that maximises the volume of the container

(ii) the maximum volume of the container. Give your answer in the form $a \times 10^k$, where $1 \leq a \leq 10$ and $k \in \mathbb{Z}$.

[4 marks]