IB Maths DP

YOUR NOTES

2. Functions

CONTENTS

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2.1 Linear Functions & Graphs

2.1.1 Equations of a Straight Line

Equations of a Straight Line

How do I find the gradient of a straight line?

- Find two points that the line passes through with coordinates (x_1, y_1) and (x_2, y_2)
- The gradient between these two points is calculated by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

  - This is given in the **formula booklet**
- The gradient of a straight line measures its **slope**
  - A line with gradient 1 will go up 1 unit for every unit it goes to the right
  - A line with gradient -2 will go down two units for every unit it goes to the right

What are the equations of a straight line?

- $y = mx + c$
  - This is the **gradient-intercept form**
  - It clearly shows the gradient *m* and the *y*-intercept (0, c)
- $y - y_1 = m(x - x_1)$
  - This is the **point-gradient form**
  - It clearly shows the gradient *m* and a point on the line (x_1, y_1)
- $ax + by + d = 0$
  - This is the **general form**
  - You can quickly get the *x*-intercept $\left(-\frac{d}{a}, 0\right)$ and *y*-intercept $\left(0, -\frac{d}{b}\right)$

How do I find an equation of a straight line?

- You will need the gradient
  - If you are given two points then first find the gradient
- It is easiest to start with the **point-gradient form**
  - then rearrange into whatever form is required
    - multiplying both sides by any denominators will get rid of fractions
- You can check your answer by using your GDC
  - Graph your answer and check it goes through the point(s)
  - If you have two points then you can enter these in the **statistics mode** and find the regression line $y = ax + b$

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Exam Tip

- A sketch of the graph of the straight line(s) can be helpful, even if not demanded by the question
  - Use your GDC to plot them
- Ensure you state equations of straight lines in the format required
  - Usually $y = mx + c$ or $ax + by + d = 0$
  - Check whether coefficients need to be integers (they usually are for $ax + by + d = 0$)

Worked Example

The line l passes through the points $(-2, 5)$ and $(6, -7)$.

Find the equation of l, giving your answer in the form $ax + by + d = 0$ where a, b and c are integers to be found.

Find the gradient between (-2,5) and (6,-7)

Formula booklet

$m = \frac{-7 - 5}{6 - -2} = -\frac{3}{2}$

Gradient formula	$m = \frac{y_2 - y_1}{x_2 - x_1}$

Use the point-gradient formula

Formula booklet

Equations of a straight line	$y - y_1 = m(x - x_1)$

$(x_1, y_1) = (-2, 5) \quad m = -\frac{3}{2}$

$y - 5 = -\frac{3}{2}(x - -2)$ Simplify

$y - 5 = -\frac{3}{2}(x + 2)$ Multiply by denominator

$2(y - 5) = -3(x + 2)$ Expand

$2y - 10 = -3x - 6$ Rearrange

$3x + 2y - 4 = 0$

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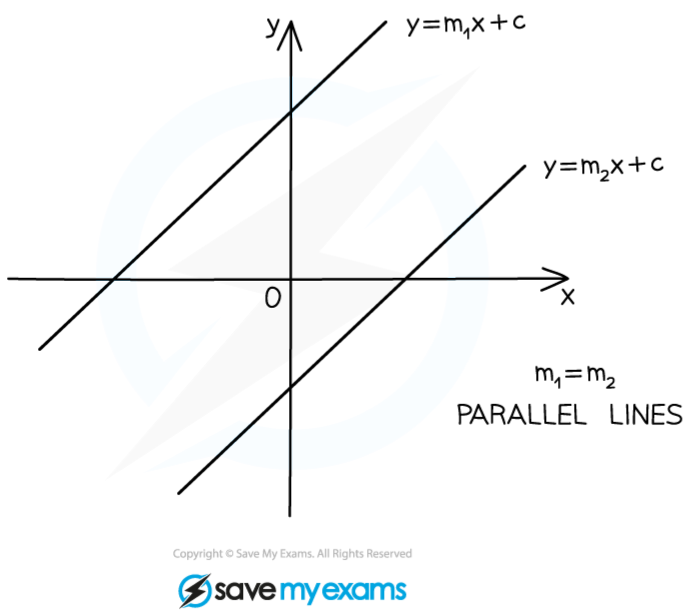
Parallel Lines

How are the equations of parallel lines connected?

- **Parallel lines** are always equidistant meaning they never intersect
- Parallel lines have the same gradient
    - If the gradient of line l_1 is m_1 and gradient of line l_2 is m_2 then...
        - $m_1 = m_2 \Rightarrow l_1$ & l_2 are parallel
        - l_1 & l_2 are parallel $\Rightarrow m_1 = m_2$
- To determine if two lines are parallel:
    - Rearrange into the gradient-intercept form $y = mx + c$
    - Compare the coefficients of x
    - If they are equal then the lines are parallel



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Worked Example

The line l passes through the point $(4, -1)$ and is parallel to the line with equation $2x - 5y = 3$.

Find the equation of l, giving your answer in the form $y = mx + c$.

Rearrange into $y = mx + c$ to find the gradient

$5y = 2x - 3 \Rightarrow y = \frac{2}{5}x - \frac{3}{5} \quad \therefore \text{gradient} = \frac{2}{5}$

Parallel lines $\Rightarrow m_1 = m_2$

$m = \frac{2}{5}$

Use the point-gradient formula

Formula booklet

Equations of a straight line	$y - y_1 = m(x - x_1)$

$(x_1, y_1) = (4, -1) \quad m = \frac{2}{5}$

$y + 1 = \frac{2}{5}(x - 4)$

$y + 1 = \frac{2}{5}x - \frac{8}{5}$

$y = \frac{2}{5}x - \frac{13}{5}$

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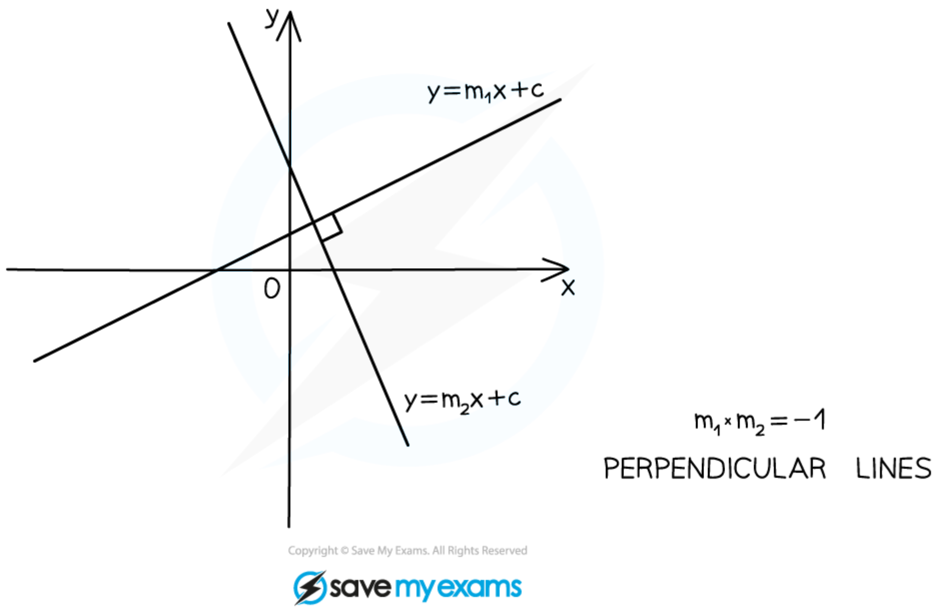
Perpendicular Lines

How are the equations of perpendicular lines connected?

- **Perpendicular lines** intersect at right angles
- The gradients of two perpendicular lines are negative reciprocals
  - If the gradient of line l_1 is m_1 and gradient of line l_2 is m_2 then...
    - $m_1 \times m_2 = -1 \Rightarrow l_1$ & l_2 are perpendicular
    - l_1 & l_2 are perpendicular $\Rightarrow m_1 \times m_2 = -1$
- To determine if two lines are perpendicular:
  - Rearrange into the gradient-intercept form $y = mx + c$
  - Compare the coefficients of x
  - If their product is -1 then they are perpendicular
- Be careful with horizontal and vertical lines
  - $x = p$ and $y = q$ are perpendicular where p and q are constants



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Worked Example

The line l_1 is given by the equation $3x - 5y = 7$.

The line l_2 is given by the equation $y = \frac{1}{4} - \frac{5}{3}x$.

Determine whether l_1 and l_2 are perpendicular. Give a reason for your answer.

Rearrange l_1 into $y = mx + c$ form

$5y = 3x - 7 \Rightarrow y = \frac{3}{5}x - \frac{7}{5}$

Identity gradients

$m_1 = \frac{3}{5}$ $\quad m_2 = -\frac{5}{3}$

$m_1 \times m_2 = -1 \Rightarrow$ Perpendicular lines

$\frac{3}{5} \times -\frac{5}{3} = -1$

l_1 and l_2 are perpendicular as $m_1 \times m_2 = -1$

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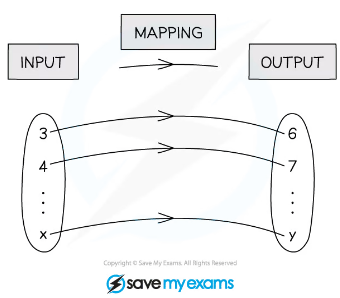
2.2 Further Functions & Graphs

2.2.1 Functions

Language of Functions

What is a mapping?

- A **mapping transforms** one set of values (**inputs**) into another set of values (**outputs**)
- Mappings can be:
  - **One-to-one**
    - Each input gets mapped to **exactly one unique** output
    - No two inputs are mapped to the same output
    - For example: A mapping that cubes the input
  - **Many-to-one**
    - Each input gets mapped to **exactly one** output
    - Multiple inputs can be mapped to the same output
    - For example: A mapping that squares the input
  - **One-to-many**
    - An input can be mapped to **more than one** output
    - No two inputs are mapped to the same output
    - For example: A mapping that gives the numbers which when squared equal the input
  - **Many-to-many**
    - An input can be mapped to **more than one** output
    - Multiple inputs can be mapped to the same output
    - For example: A mapping that gives the factors of the input



What is a function?

- A **function** is a mapping between two sets of numbers where **each input** gets mapped to **exactly one output**
  - The output does not need to be unique
- **One-to-one** and **many-to-one** mappings are functions
- A mapping is a function if its graph passes the **vertical line test**
  - Any **vertical line** will intersect with the graph **at most once**

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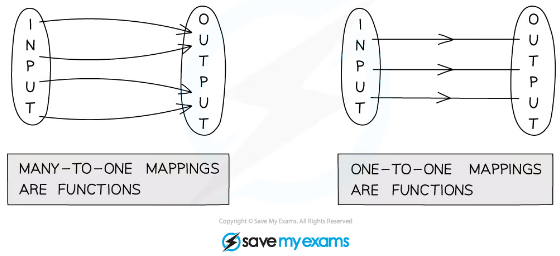

What notation is used for functions?

- Functions are denoted using letters (such as f, v, g, etc)
  - A function is followed by a variable in a bracket
  - This shows the input for the function
  - The letter f is used most commonly for functions and will be used for the remainder of this revision note
- $f(x)$ represents an expression for the value of the function f when evaluated for the variable x
- Function notation gets rid of the need for words which makes it **universal**
  - $f = 5$ when $x = 2$ can simply be written as $f(2) = 5$

What are the domain and range of a function?

- The **domain** of a function is the set of values that are used as **inputs**
- A domain should be stated with a function
  - If a domain is not stated then it is assumed the domain is all the real values which would work as inputs for the function
  - Domains are expressed in terms of the input
    - $x \leq 2$
- The **range** of a function is the set of values that are given as **outputs**
  - The range depends on the domain
  - Ranges are expressed in terms of the output
    - $f(x) \geq 0$
- To graph a function we use the **inputs as the *x*-coordinates** and the **outputs as the *y*-coordinates**
  - $f(2) = 5$ corresponds to the coordinates (2, 5)
- Graphing the function can help you visualise the range
- Common sets of numbers have special symbols:
  - $\mathbb{R}$ represents all the real numbers that can be placed on a number line
    - $x \in \mathbb{R}$ means x is a real number
  - $\mathbb{Q}$ represents all the rational numbers $\frac{a}{b}$ where *a* and *b* are integers and $b \neq 0$
  - $\mathbb{Z}$ represents all the integers (positive, negative and zero)
    - $\mathbb{Z}^+$ represents positive integers
  - $\mathbb{N}$ represents the natural numbers (0,1,2,3...)

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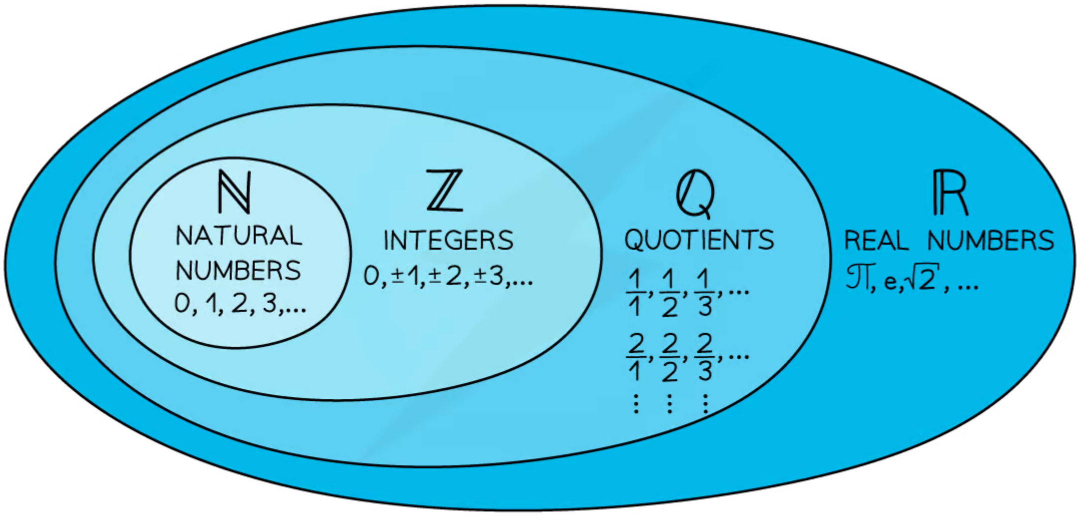

Exam Tip

- Questions may refer to "the largest possible domain"
  - this would usually be $x \in \mathbb{R}$ unless natural numbers, integers or quotients has already been stated
  - there are usually some exceptions
    - e.g. square roots; $x \geq 0$ for a function involving $\sqrt{x}$
    - e.g. reciprocal functions; $x \neq 2$ for a function with denominator $(x - 2)$

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Worked Example

For the function $f(x) = x^3 + 1,\ \ 2 \le x \le 10$:

a)
write down the value of $f(7)$.

Substitute $x = 7$

$f(7) = 7^3 + 1$

$f(7) = 344$

b)
find the range of $f(x)$.

Find the values of x^3+1 when $2 \le x \le 10$

$2 \le x \le 10$

$8 \le x^3 \le 1000$

$9 \le x^3+1 \le 1001$

$9 \le f(x) \le 1001$

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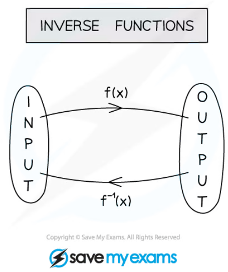
Inverse Functions

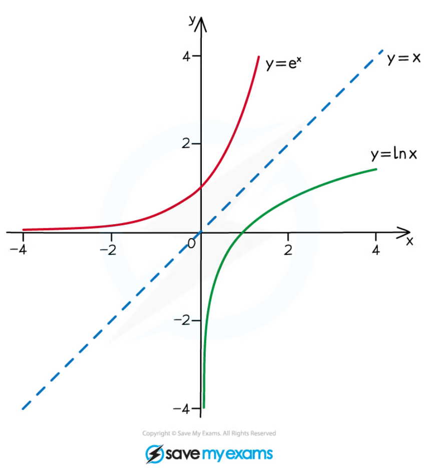
What is an inverse function?

- **Only one-to-one** functions have inverses
- A function has an inverse if its graph passes the **horizontal line test**
  - Any **horizontal line** will intersect with the graph **at most once**
- Given a function $f(x)$ we denote the inverse function as $f^{-1}(x)$
- An inverse function **reverses the effect** of a function
  - $f(2)=5$ means $f^{-1}(5)=2$
- Inverse functions are used to solve equations
  - The solution of $f(x)=5$ is $x=f^{-1}(5)$



What are the connections between a function and its inverse function?

- The **domain of a function** becomes the **range of its inverse**
- The **range of a function** becomes the **domain of its inverse**
- The graph of $y=f^{-1}(x)$ is a **reflection** of the graph $y=f(x)$ in the line $y=x$
  - Therefore solutions to $f(x)=x$ or $f^{-1}(x)=x$ will also be solutions to $f(x)=f^{-1}(x)$
    - There could be other solutions to $f(x)=f^{-1}(x)$ that don't lie on the line $y=x$



Exam Tip

- Remember that, in general, $f^{-1}(x) \neq \frac{1}{f(x)}$

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Worked Example

For the function $f(x) = x^3 + 1,\ 2 \le x \le 10$:

a)
write down the range of the inverse function, $f^{-1}(x)$.

The range of $f^{-1}(x)$ is the domain of $f(x)$

$2 \le f^{-1}(x) \le 10$

b)
find the value of $f^{-1}(217)$.

If $x = f^{-1}(a)$ then $f(x) = a$

$x = f^{-1}(217)$

$f(x) = 217$

$x^3 + 1 = 217$ (Use definition of function)

$x^3 = 216$ (Subtract 1)

$x = 6$ (Cube root)

$f^{-1}(217) = 6$

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Piecewise Functions

What are piecewise functions?

- **Piecewise functions** are defined by different functions depending on which interval the input is in
  - E.g. $f(x)=\begin{cases} x+1 & x\le 5 \\ 2x-4 & 5<x<10 \end{cases}$
- The region for the individual functions **cannot overlap**
- To evaluate a piecewise function for a particular value $x=k$
  - Find which interval includes k
  - Substitute $x=k$ into the corresponding function

Worked Example

For the piecewise function

$$f(x)=\begin{cases} 2x-5 & -10\le x\le 10 \\ 3x+1 & x>10 \end{cases},$$

a)
find the values of $f(0)$, $f(10)$, $f(20)$.

Identify the correct function to use

$x=0$ is in $-10\le x\le 10$ ⇒ $f(0)=2(0)-5=-5$

$x=10$ is in $-10\le x\le 10$ ⇒ $f(10)=2(10)-5=15$

$x=20$ is in $x>10$ ⇒ $f(20)=3(20)+1=61$

$f(0)=-5$ $f(10)=15$ $f(20)=61$

b)
state the domain.

Domain is the set of inputs

$-10\le x\le 10$ and $x>10$

$x\ge -10$

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Graphing Functions

How do I graph the function *y* = *f*(*x*)?

- A point (a, b) lies on the graph $y = f(x)$ if $f(a) = b$
- The **horizontal axis** is used for the **domain**
- The **vertical axis** is used for the **range**
- You will be able to graph some functions by hand
- For some functions you will need to use your GDC
- You might be asked to graph the **sum** or **difference** of two functions
    - Use your GDC to graph $y = f(x) + g(x)$ or $y = f(x) - g(x)$
    - Just type the functions into the graphing mode

What is the difference between "draw" and "sketch"?

- If asked to sketch you should:
    - Show the general shape
    - Label any key points such as the intersections with the axes
    - Label the axes
- If asked to draw you should:
    - Use a pencil and ruler
    - Draw to scale
    - Plot any points **accurately**
    - Join points with a straight line or smooth curve
    - Label any key points such as the intersections with the axes
    - Label the axes

How can my GDC help me sketch/draw a graph?

- You use your GDC to plot the graph
    - Check the scales on the graph to make sure you see the full shape
- Use your GDC to find any key points
- Use your GDC to check specific points to help you plot the graph

Key Features of Graphs

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What are the key features of graphs?

- You should be familiar with the following key features and know how to use your GDC to find them
- Local minimums/maximums
  - These are points where the graph has a minimum/maximum for a small region
  - They are also called **turning points**
    - This is where the graph changes its direction between upwards and downwards directions
  - A graph can have multiple local minimums/maximums
  - A local minimum/maximum is not necessarily the minimum/maximum of the whole graph
    - This would be called the **global** minimum/maximum
  - For quadratic graphs the minimum/maximum is called the **vertex**
- Intercepts
  - y – intercepts are where the graph crosses the y-axis
    - At these points $x = 0$
  - x – intercepts are where the graph crosses the x-axis
    - At these points $y = 0$
    - These points are also called the **zeros of the function** or **roots of the equation**
- Symmetry
  - Some graphs have lines of symmetry
    - A quadratic will have a vertical line of symmetry
- Asymptotes
  - These are lines which the graph will get closer to but not cross
  - These can be horizontal or vertical
    - Exponential graphs have horizontal asymptotes
    - Graphs of variables which vary inversely can have vertical and horizontal asymptotes

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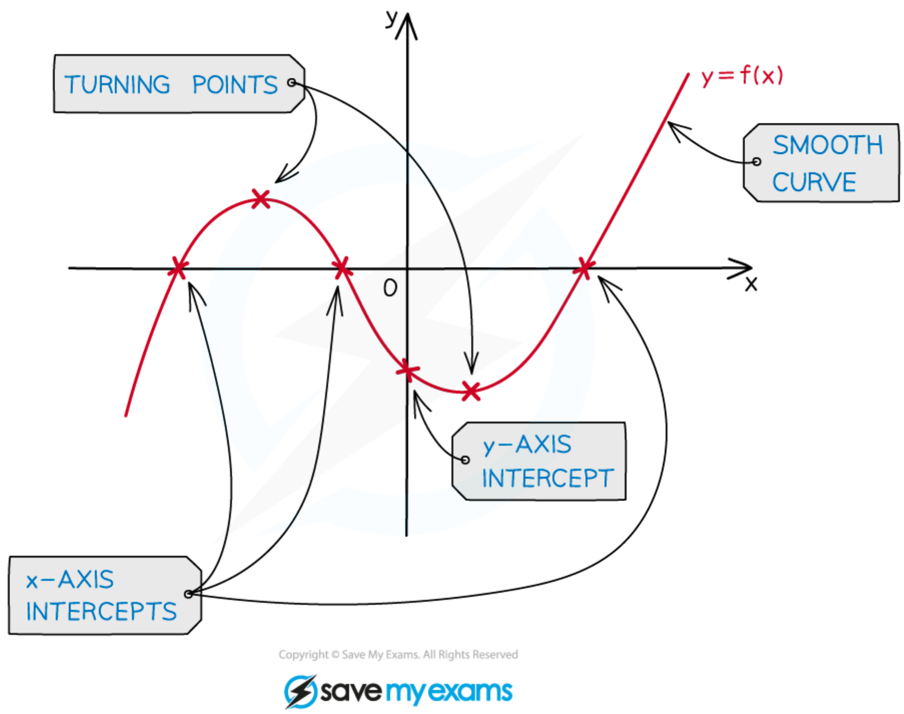

Exam Tip

- Most GDC makes/models will not plot/show asymptotes just from inputting a function
  - Add the asymptotes as additional graphs for your GDC to plot
  - You can then check the equations of your asymptotes visually
  - You may have to zoom in or change the viewing window options to confirm an asymptote
- Even if using your GDC to plot graphs and solve problems sketching them as part of your working is good exam technique
  - Label the key features of the graph and anything else relevant to the question on your sketch

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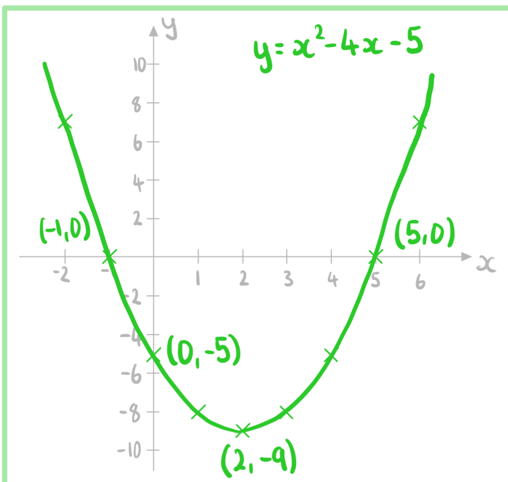
Worked Example

Two functions are defined by

$$f(x) = x^2 - 4x - 5 \text{ and } g(x) = 2 + \frac{1}{x+1}.$$

a)
Draw the graph $y = f(x)$.

Draw means accurately
Use GDC to find vertex, roots and y-intercepts
Vertex : (2, -9)
Roots : (-1, 0) and (5, 0)
y-intercept : (0, -5)



b)
Sketch the graph $y = g(x)$.

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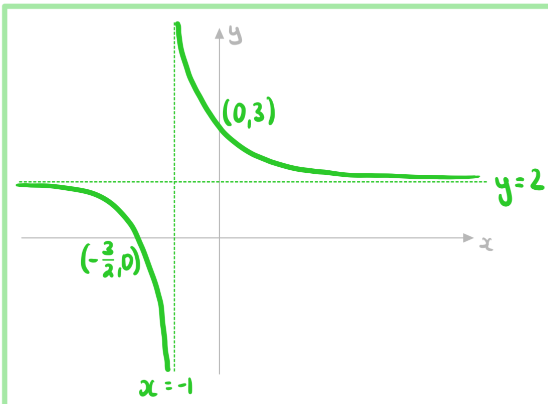
Sketch means rough but showing key points

Use GDC to find x and y-intercepts and asymptotes

x-intercept $= \left(-\frac{3}{2}, 0\right)$

y-intercept $= (0, 3)$

Asymptotes : $x = -1$ and $y = 2$



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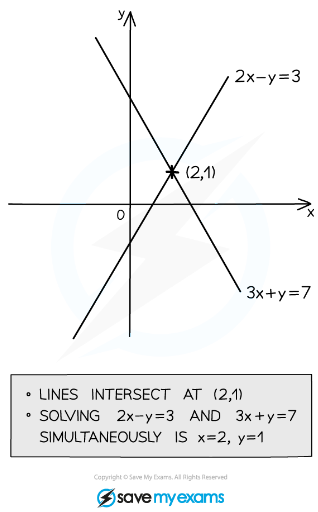
Intersecting Graphs

How do I find where two graphs intersect?

- Plot both graphs on your GDC
- Use the intersect function to find the intersections
- Check if there is more than one point of intersection



How can I use graphs to solve equations?

- One method to solve equations is to use graphs
- To solve $f(x) = a$
  - Plot the two graphs $y = f(x)$ and $y = a$ on your GDC
  - Find the points of intersections
  - The ***x*-coordinates** are the **solutions** of the equation
- To solve $f(x) = g(x)$
  - Plot the two graphs $y = f(x)$ and $y = g(x)$ on your GDC
  - Find the points of intersections
  - The ***x*-coordinates** are the **solutions** of the equation
- Using graphs makes it easier to see **how many solutions** an equation will have

Exam Tip

- You can use graphs to solve equations
  - Questions will not necessarily ask for a drawing/sketch or make reference to graphs
  - Use your GDC to plot the equations and find the intersections between the graphs

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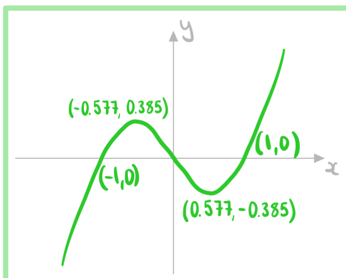
Worked Example

Two functions are defined by

$$f(x) = x^3 - x \text{ and } g(x) = \frac{4}{x}.$$

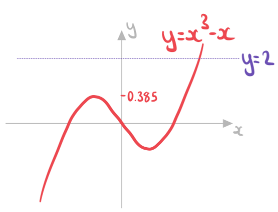
a)
Sketch the graph $y = f(x)$.

Use GDC to find max, min, intercepts



b)
Write down the number of real solutions to the equation $x^3 - x = 2$.

Identify the number of intersections between
$y = x^3 - x$ and $y = 2$



1 intersection

1 solution

c)
Find the coordinates of the points where $y = f(x)$ and $y = g(x)$ intersect.

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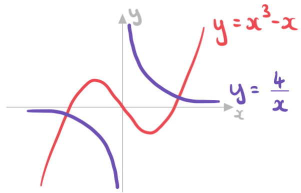
Use GDC to sketch both graphs



(-1.60, -2.50) and (1.60, 2.50)

d)

Write down the solutions to the equation $x^3 - x = \frac{4}{x}$.

Solutions to $x^3 - x = \frac{4}{x}$ are the x coordinates of the points of intersection.

$x = -1.60$ and $x = 1.60$

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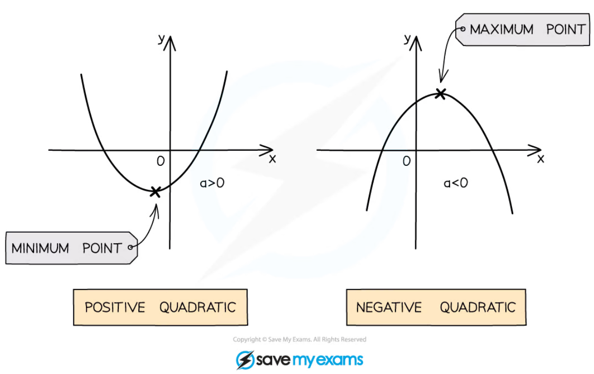
Quadratic Functions & Graphs

What are the key features of quadratic graphs?

- A **quadratic** graph is of the form $y = ax^2 + bx + c$ where $a \neq 0$.
- The value of *a* affects the shape of the curve
  - If *a* is positive the shape is ∪
  - If *a* is negative the shape is ∩
- The ***y*-intercept** is at the point (0, c)
- The **zeros or roots** are the solutions to $ax^2 + bx + c = 0$
  - These can be found using your GDC or the quadratic formula
  - These are also called the *x*-intercepts
  - There can be 0, 1 or 2 *x*-intercepts
- There is an **axis of symmetry** at $x = -\frac{b}{2a}$
  - This is given in your **formula booklet**
  - If there are two *x*-intercepts then the axis of symmetry goes through the midpoint of them
- The **vertex** lies on the axis of symmetry
  - The *x*-coordinate is $-\frac{b}{2a}$
  - The *y*-coordinate can be found using the GDC or by calculating *y* when $x = -\frac{b}{2a}$
  - If ***a* is positive** then the vertex is the **minimum** point
  - If ***a* is negative** then the vertex is the **maximum** point



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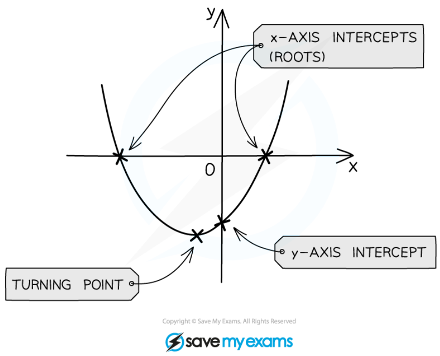

Exam Tip

- Use your GDC to find the roots and the turning point of a quadratic function
  - You do not need to factorise or complete the square
  - It is good exam technique to sketch the graph from your GDC as part of your working

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Worked Example

a)
Write down the equation of the axis of symmetry for the graph $y = 4x^2 - 4x - 3$.

Formula booklet

Axis of symmetry of the graph of a quadratic function	$f(x) = ax^2 + bx + c \Rightarrow$ axis of symmetry is $x = -\frac{b}{2a}$

$a = 4 \quad b = -4 \quad c = -3$

$x = -\frac{-4}{2(4)}$

$x = \frac{1}{2}$

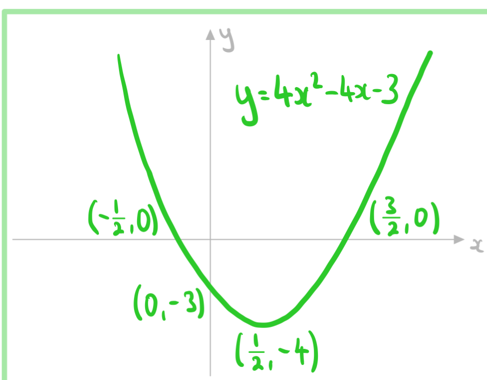
b)
Sketch the graph $y = 4x^2 - 4x - 3$.

Use GDC to find vertex, roots and y-intercepts

Vertex = $(\frac{1}{2}, -4)$

Roots = $(-\frac{1}{2}, 0)$ and $(\frac{3}{2}, 0)$

y-intercept = $(0, -3)$



YOUR NOTES
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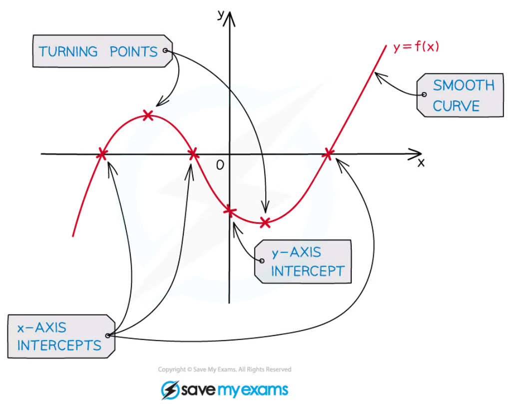
Cubic Functions & Graphs

What are the key features of cubic graphs?

- A **cubic** graph is of the form $y = ax^3 + bx^2 + cx + d$ where $a \neq 0$.
- The value of *a* affects the shape of the curve
  - If ***a* is positive** the graph goes from **bottom left to top right**
  - If ***a* is negative** the graph goes from **top left to bottom right**
- The ***y*-intercept** is at the point (0, *d*)
- The **zeros or roots** are the solutions to $ax^3 + bx^2 + cx + d = 0$
  - These can be found using your GDC
  - These are also called the *x*-intercepts
  - There can be 1, 2 or 3 *x*-intercepts
    - There is always at least 1
- There are either **0 or 2 local minimums/maximums**
  - If there are 0 then the curve is **monotonic** (always increasing or always decreasing)
  - If there are 2 then one is a local minimum and one is a local maximum



Exam Tip

- You can use your GDC to find the roots, the local maximum and local minimum of a cubic function
- When drawing/sketching the graph of a cubic function be sure to label all the key features
  - x and y axes intercepts
  - the local maximum point
  - the local minimum point

YOUR NOTES
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Worked Example

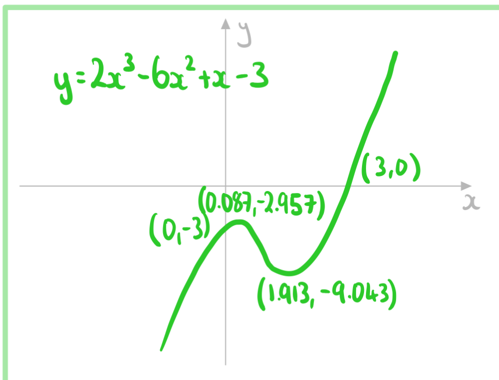
Sketch the graph $y = 2x^3 - 6x^2 + x - 3$.

Use GDC to find min, max, roots and y-intercept

Min = (1.913, -9.043) Max = (0.087, -2.957)

Root = (3,0)

y-intercept = (0, -3)



YOUR NOTES
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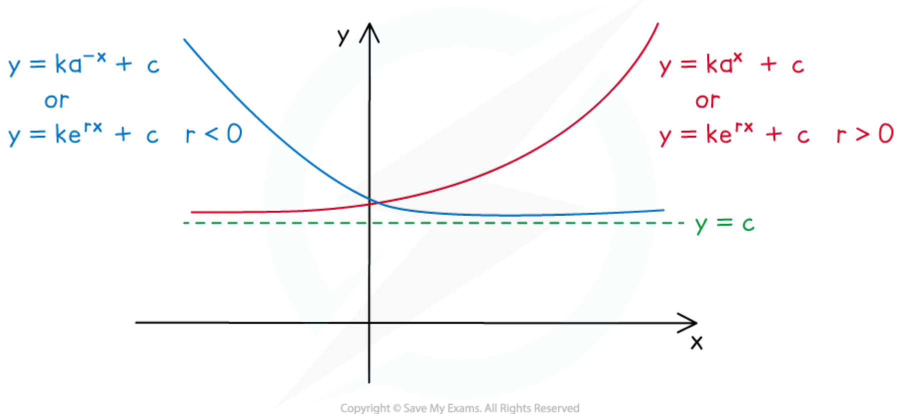
Exponential Functions & Graphs

What are the key features of exponential graphs?

- An **exponential** graph is of the form
  - $y = ka^x + c$ or $y = ka^{-x} + c$ where $a > 0$
  - $y = ke^{rx} + c$
    - Where e is the mathematical constant 2.718…
- The ***y*-intercept** is at the point (0, *k* + c)
- There is a **horizontal asymptote** at *y* = c
- The value of *k* determines whether the graph is **above or below the asymptote**
  - If ***k* is positive** the graph is **above the asymptote**
    - So the range is $y > c$
  - If ***k* is negative** the graph is **below the asymptote**
    - So the range is $y < c$
- The coefficient of *x* and the constant *k* determine whether the graph is **increasing or decreasing**
  - If the coefficient of *x* and *k* have the **same sign** then **graph is increasing**
  - If the coefficient of *x* and *k* have **different signs** then the **graph is decreasing**
- There is at **most 1 root**
  - It can be found using your GDC



Exam Tip

- You may have to change the viewing window settings on your GDC to make asymptotes clear
  - A small scale can make it look as though the curve and an asymptote intercept
- Be careful about how two exponential graphs drawn on the same axes look
  - Particularly which one is "on top" either side of the y-axis

YOUR NOTES
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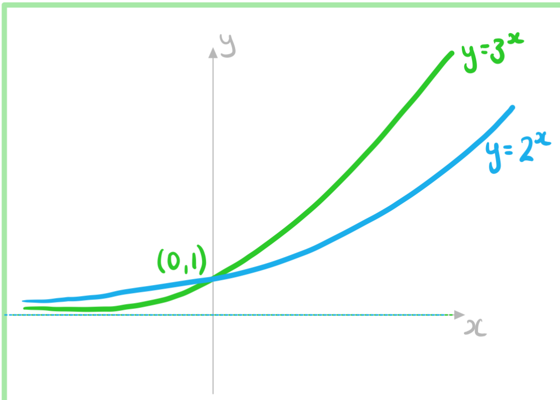
Worked Example

a)
On the same set of axes sketch the graphs $y = 2^x$ and $y = 3^x$. Clearly label each graph.

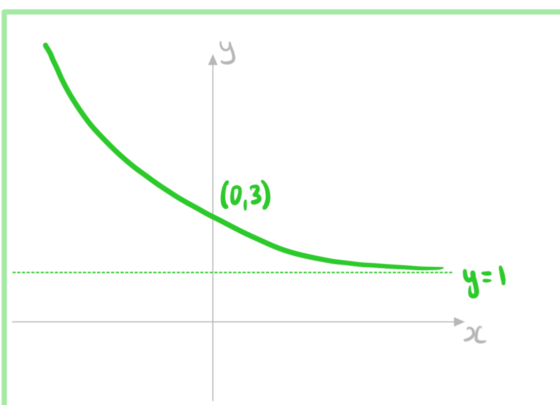


b)
Sketch the graph $y = 2e^{-3x} + 1$.

Use GDC to find intercept and asymptote

y-intercept = (0, 3)

Asymptote : y = 1



YOUR NOTES
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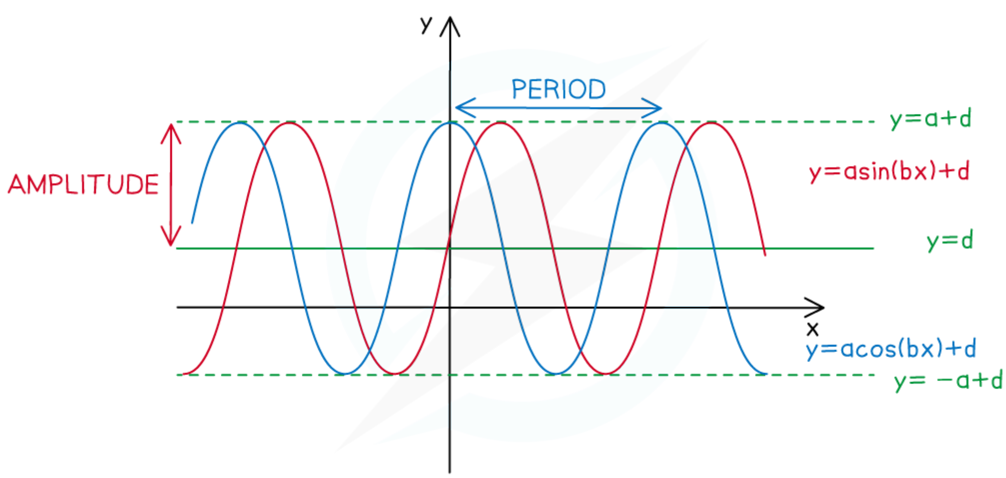
Sinusoidal Functions & Graphs

What are the key features of sinusoidal graphs?

- A **sinusoidal** graph is of the form
  - $y = a\sin(bx) + d$
  - $y = a\cos(bx) + d$
- The ***y*-intercept** is at the point
  - $(0, d)$ for $y = a\sin(bx) + d$
  - $(0, a + d)$ for $y = a\cos(bx) + d$
- The **period** of the graph is the length of the interval of a full cycle
  - This is $\frac{360°}{b}$
- The **maximum value** is $y = a + d$
- The **minimum value** is $y = -a + d$
- The **principal axis** is the horizontal line halfway between the maximum and minimum values
  - This is $y = d$
- The **amplitude** is the vertical distance from the principal axis to the maximum value
  - This is a



Exam Tip

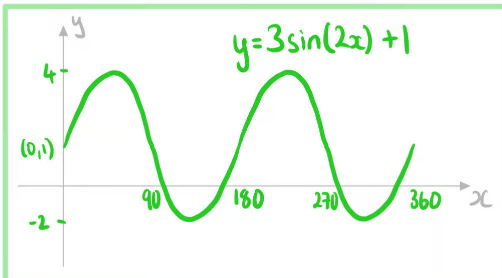
- Pay careful attention to the angles between which you are required to use or draw/sketch a sinusoidal graph
  - e.g. $0° \leq x \leq 360°$

YOUR NOTES ↓

Worked Example

a)
Sketch the graph $y = 3\sin(2x) + 1$ for the values $0 \le x \le 360$.

Use GDC to find max and min



b)
State the equation of the principal axis of the curve.

Principal axis is in middle of maximum and minimum points

$\frac{4 + {}^{-}2}{2} = 1$

$y = 1$

c)
State the period and amplitude.

Period is how often it repeats

$\frac{360}{2} = 180$

Period $= 180°$

Amplitude is distance from principal axis to maximum or minimum

$4 - 1 = 1 - {}^{-}2 = 3$

Amplitude $= 3$

YOUR NOTES
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2.3 Modelling with Functions

2.3.1 Linear & Piecewise Models

Linear Models

What are the parameters of a linear model?

- A **linear model** is of the form $f(x) = mx + c$
- The *m* represents the **rate of change** of the function
  - This is the amount the function increases/decreases when *x* increases by 1
    - If the function is increasing *m* is positive
    - If the function is decreasing *m* is negative
  - When the model is represented as a graph this is the **gradient** of the line
- The *c* represents the value of the function when $x = 0$
  - This is the value of the function when the independent variable is not present
  - This is usually referred to as the initial value
  - When the model is represented as a graph this is the **y-intercept** of the line

What can be modelled as a linear model?

- If the graph of the data resembles a **straight line**
- Anything with a **constant** rate of change
  - $C(d)$ is the taxi charge for a journey of *d* km
  - $B(m)$ is the monthly mobile phone bill when *m* minutes have been used
  - $R(d)$ is the rental fee for a car used for *d* days
  - $d(t)$ is the distance travelled by a car moving at a constant speed for *t* seconds

What are possible limitations of a linear model?

- Linear models continuously increase (or decrease) at the same rate
  - In real-life this might not be the case
  - The function might reach a maximum (or minimum)
- If the value of *m* is negative then for some inputs the function will predict negative values
  - In some real-life situations negative values will not make sense
  - To overcome this you can decide on an appropriate domain so that the outputs are never negative

Exam Tip

- Make sure that you are equally confident in working with linear models both algebraically and graphically as it may be easier using one method over the other when tackling a particular exam question

YOUR NOTES
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Worked Example

The total cost, C, in New Zealand dollars (NZD), of a premium gym membership at FitFirst can be modelled by the function

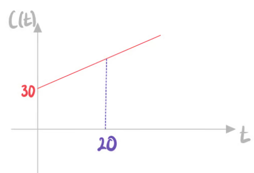
$$C = 14.95t + 30, \quad t \geq 0$$

where t is the time in weeks.

a)
Calculate the cost of the gym membership for 20 weeks.

Substitute $t = 20$

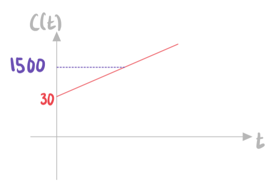
$C(20) = 14.95(20) + 30$



329 NZD

b)
Find the number of weeks it takes for the total cost to exceed 1500 NZD.

Substitute $C = 1500$



$1500 = 14.95t + 30$

$14.95t = 1470$

$t = \frac{1470}{14.95} = 98.327...$

Round up to the next integer

99 weeks

c)
Under new management, FitFirst changes the initial payment to 20 NZD and the weekly cost to 19.25 NZD. Write the new cost function after these changes have been.

$C(t) = mt + c$

m is the constant rate per week $m = 19.25$

c is the initial cost $c = 20$

$C(t) = 19.25t + 20$

YOUR NOTES
⬇

Linear Piecewise Models

What are the parameters of a piecewise linear model?

- A **piecewise linear model** is made up of multiple linear models $f_i(x) = m_i x + c_i$
- For each linear model there will be
  - The rate of change for that interval m_i
  - The value if the independent variable was not present c_i

What can be modelled as a piecewise linear model?

- Piecewise linear models can be used when the rate of change of a function changes for different intervals
  - These commonly apply when there are different tariffs or levels of charges
- Anything with a constant rate of change for set intervals
  - $C(d)$ is the taxi charge for a journey of d km
    - The charge might double after midnight
  - $R(d)$ is the rental fee for a car used for d days
    - The daily fee might triple if the car is rented over bank holidays
  - $s(t)$ is the speed of a car travelling for t seconds with constant acceleration
    - The car might reach a maximum speed

What are possible limitations of a piecewise linear model?

- Linear models have a constant rate of change
  - In real-life this might not be the case
  - A function might increase (or decrease) gradually rather than at a constant rate

Exam Tip

- Make sure that you know how to plot a piecewise model on your GDC

YOUR NOTES

Worked Example

The total monthly charge, £ C, of phone bill can be modelled by the function

$$C(m) = \begin{cases} 10 + 0.02m & 0 \le m \le 100 \\ 9 + 0.03m & m > 100 \end{cases},$$

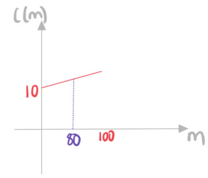
where m is the number of minutes used.

a)
Find the total monthly charge if 80 minutes have been used.

Substitute $m = 80$ into the first function

$C(80) = 10 + 0.02(80)$

£11.60



b)
Given that the total monthly charge is £16.59, find the number of minutes that were used.

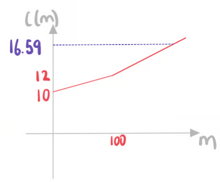
Substitute $C = 16.59$ into the second function

$16.59 = 9 + 0.03m$

$0.03m = 7.59$

$m = \frac{7.59}{0.03}$

253 minutes



2.3.2 Quadratic & Cubic Models

YOUR NOTES
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Quadratic Models

What are the parameters of a quadratic model?

- A quadratic model is of the form $f(x) = ax^2 + bx + c$
- The c represents the value of the function when $x = 0$
  - This is the value of the function when the independent variable is not present
  - This is usually referred to as the initial value
- The *a* has the biggest impact on the rate of change of the function
  - If *a* has a large absolute value then the rate of change varies rapidly
  - If *a* has a small absolute value then the rate of change varies slowly
- The maximum (or minimum) of the function occurs when $x = -\frac{b}{2a}$
  - This is given in the **formula booklet** as the **axis of symmetry**

What can be modelled as a quadratic model?

- If the graph of the data resembles a ∪ or ∩ shape
- These can be used if the graph has a single maximum or minimum
  - $H(t)$ is the vertical height of a football t seconds after being kicked
  - $A(x)$ is the area of rectangle of length x cm that can be made with a 20 cm length of string

What are possible limitations of a quadratic model?

- A quadratic has either a maximum or a minimum but **not both**
  - This means one end is **unbounded**
  - In real-life this might not be the case
  - The function might have both a maximum and a minimum
  - To overcome this you can decide on an appropriate domain so that the outputs are within a range
- Quadratic graphs are **symmetrical**
  - This might not be the case in real-life

Exam Tip

- Read and re-read the question carefully, try to get involved in the context of the question!
  - Imagine what happens to a stone as you throw it from a cliff, what would the path look like?
  - What would it be like to manage a toy factory, would you expect profit to rise or fall as you increase the price of the toy?
- **Sketch** a graph of the function being used as the model, use your GDC to help you
- If you are completely stuck try "doing something" with the quadratic function – sketch it, factorise it, solve it

YOUR NOTES

Worked Example

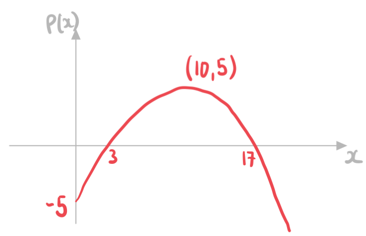
A company sells unicorn toys. The profit, £ P, of the selling one unicorn toy can be modelled by the function

$$P(x) = \frac{1}{10}(-x^2 + 20x - 50)$$

where x is the selling price of the toy.

Find the selling price which maximises profit. State the maximum profit.

Sketch on GDC and find the maximum point



Selling price £10
Maximum profit £5

YOUR NOTES
⬇

Cubic Models

What are the parameters of a cubic model?

- A **cubic model** is of the form $f(x) = ax^3 + bx^2 + cx + d$
- The *d* represents the value of the function when $x = 0$
  - This is the value of the function when the independent variable is not present
  - This is usually referred to as the initial value
- The *a* has the biggest impact on the rate of change of the function
  - If *a* has a large absolute value then the rate of change varies rapidly
  - If *a* has a small absolute value then the rate of change varies slowly

What can be modelled as a cubic model?

- If the graph of the data has exactly one maximum and one minimum within an interval
- If the graph is monotonic with no maximum or minimum
  - $D(t)$ is the vertical distance below starting point of a bungee jumper *t* seconds after jumping
  - $V(x)$ is the volume of a cuboid of length *x* cm that can be made with a 200 cm^2 of cardboard

What are possible limitations of a cubic model?

- Cubic graphs have **no global maximum or minimum**
  - This means the function is **unbounded**
  - In real-life this might not be the case
  - The function might have a maximum or minimum
  - To overcome this you can decide on an appropriate domain so that the outputs are within a range

Exam Tip

- Read and re-read the question carefully, try to get involved in the context of the question!
- Always sketch the graph using your GDC to help
- Pay particular attention to the domain of the question
  - If the domain is given, make sure that you focus only on that section when you sketch the graph
  - If the domain is not given, think about whether or not it needs to be restricted based on the context of the question, e.g. can time be negative?

YOUR NOTES
⬇

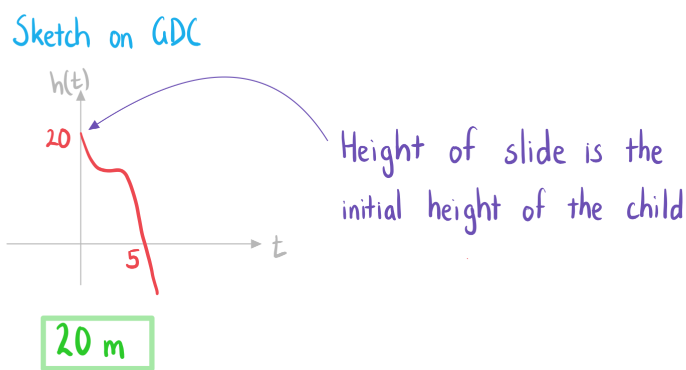
Worked Example

The vertical height of a child above the ground, h metres, as they go down a water slide can be modelled by the function

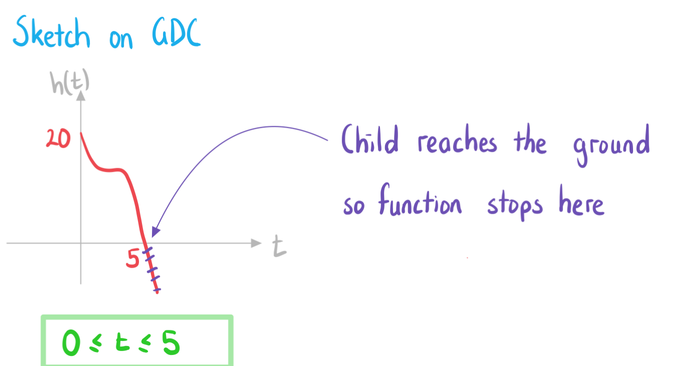
$$h(t) = \frac{4}{7}(35 - 12t + 6t^2 - t^3),$$

where t is the time in seconds after the child enters the slide.

a)
State the vertical height of the slide.



b)
Given that the child reaches the ground at the bottom of the slide, find the domain of the function.



2.3.3 Exponential Models

YOUR NOTES

Exponential Models

What are the parameters of an exponential model?

- An **exponential model** is of the form
  - $f(x) = ka^x + c$ or $f(x) = ka^{-x} + c$ for $a > 0$
  - $f(x) = ke^{rx} + c$
    - Where e is the mathematical constant 2.718…
  - The c represents the **boundary** for the function
    - It can never be this value
  - The *a* or *r* describes the **rate of growth or decay**
    - The bigger the value of *a* or the absolute value of *r* the faster the function increases/decreases

What can be modelled as an exponential model?

- Exponential growth or decay
  - Exponential **growth** is represented by
    - a^x where $a > 1$
    - a^{-x} where $0 < a < 1$
    - e^{rx} where $r > 0$
  - Exponential **decay** is represented by
    - a^x where $0 < a < 1$
    - a^{-x} where $a > 1$
    - e^{rx} where $r < 0$
- They can be used when there a **constant percentage increase or decrease**
  - Such as functions generated by **geometric sequences**
- Examples include:
  - *V*(*t*) is the value of car after *t* years
  - *S*(*t*) is the amount in a savings account after *t* years
  - *B*(*t*) is the amount of bacteria on a surface after *t* seconds
  - *T*(*t*) is the temperature of a kettle *t* minutes after being boiled

What are possible limitations of an exponential model?

- An exponential growth model does not have a maximum
  - In real-life this might not be the case
  - The function might reach a maximum and stay at this value
- Exponential models are **monotonic**
  - In real-life this might not be the case
  - The function might **fluctuate**

YOUR NOTES
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Exam Tip

- Look out for the word "initial" or similar, as way of asking you to make the power equal to zero to simplify the equation
- Questions regarding the boundary of the exponential model are also frequently asked

Worked Example

The value of a car, V (NZD), can be modelled by the function

$$V(t) = 25125 \times 0.8^{t} + 8500, \quad t \geq 0$$

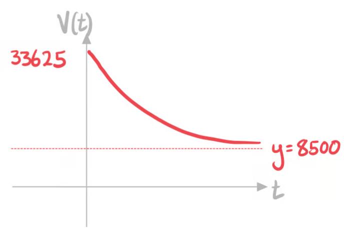
where t is the age of the car in years.

a)
State the initial value of the car.

Initial value is when $t = 0$

$V(0) = 25125 \times 0.8^{0} + 8500$

33625 NZD



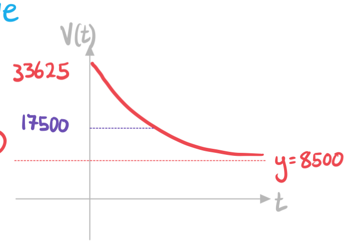
b)
Find the age of the car when its value is 17500 NZD.

Set $V(t) = 17500$ and solve on GDC

$25125 \times 0.8^{t} + 8500 = 17500$

$t = 4.6007...$

4.60 years



2.3.4 Direct & Inverse Variation

YOUR NOTES
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Direct Variation

What is direct variation?

- Two variables are said to **vary directly** if their **ratio is constant** (*k*)
  - This is also called **direct proportion**
- If y and x^n (for positive integer *n*) vary **directly** then:
  - It is denoted as $y \propto x^n$
  - $y = kx^n$ for some constant *k*
    - This can be written as $\frac{y}{x^n} = k$
- The graphs of these models always **start at the origin**

How do I solve direct variation problems?

- Identify which two variables vary directly
  - It might not be x and y
  - It could be x^3 and y
- Use the given information to find their **constant ratio *k***
  - Also called **constant of proportionality**
  - **Substitute** the given values of x and y into your formula
  - **Solve** to find *k*
- Write the equation which models their relationship
  - $y = kx^n$
- You can then use the equation to solve problems

YOUR NOTES

Worked Example

A computer program sorts a list of numbers into ascending order. The time it takes, t milliseconds, varies directly with the square of the number of items, n, in the list. The computer program takes 48 milliseconds to order a list with 8 items.

a)
Find an equation connecting t and n.

Identify the variables that vary directly

$t \propto n^2$

Form an equation

$t = kn^2$

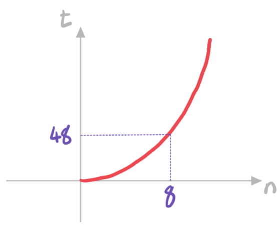
Use $t = 48$ and $n = 8$ to find the value of k

$48 = k(8)^2$

$64k = 48$

$k = \frac{48}{64} = 0.75$

$t = 0.75n^2$

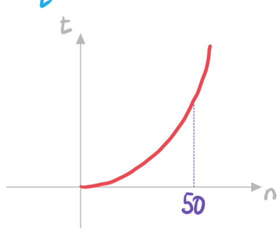


b)
Find the time it takes to order a list of 50 numbers.

Substitute $n = 50$ into the equation

$t = 0.75(50)^2$

1875 milliseconds



Inverse Variation

YOUR NOTES
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What is inverse variation?

- Two variables are said to **vary inversely** if their **product is constant (*k*)**
  - This is also called **inverse proportion**
- If y and x^n (for positive integer *n*) vary **inversely** then:
  - It is denoted $y \propto \frac{1}{x^n}$
  - $y = \frac{k}{x^n}$ for some constant *k*
    - This can be written $x^n y = k$
- The graphs of these models all have a **vertical asymptote** at the ***y*-axis**
  - This means that as x gets closer to 0 the absolute value of y gets further away from 0
  - x can never equal 0

How do I solve inverse variation problems?

- Identify which two variables vary inversely
  - It might not be x and y
  - It could be x^3 and y
- Use the given information to find their **constant product *k***
  - Also called **constant of proportionality**
  - **Substitute** the given values of x and y into your formula
  - **Solve** to find *k*
- Write the equation which models their relationship
  - $y = \frac{k}{x^n}$
- You can then use the equation to solve problems

Exam Tip

- Reciprocal graphs generally have two parts/curves
  - Only one – usually the positive – may be relevant to the model
  - Think about why **x/t/θ** can only take positive values - refer to the context of the question

YOUR NOTES
⬇

Worked Example

The time, t hours, it takes to complete a project varies inversely to the number of people working on it, n. If 4 people work on the project it takes 70 hours to complete.

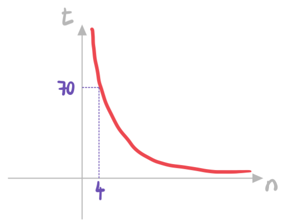
a)
Write an equation connecting t and n.

Identify the variables that vary directly

$t \propto \frac{1}{n}$

Form an equation

$t = \frac{k}{n}$



Use $t=70$ and $n=4$ to find the value of k

$70 = \frac{k}{4}$

$k = 4 \times 70 = 280$

$t = \frac{280}{n}$

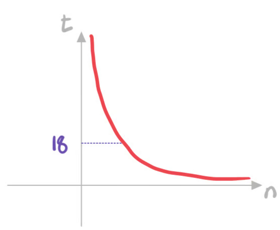
b)
Given that the project needs to be completed within 18 hours, find the minimum number of people needed to work on it.

Substitute $t=18$ into the equation

$18 = \frac{280}{n}$

$n = \frac{280}{18} = 15.55...$

16 people



2.3.5 Sinusoidal Models

YOUR NOTES
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Sinusoidal Models

What are the parameters of a sinusoidal model?

- A **sinusoidal model** is of the form
  - $f(x) = a\sin(bx) + d$
  - $f(x) = a\cos(bx) + d$
- The *a* represents the **amplitude** of the function
  - The bigger the value of *a* the bigger the **range** of values of the function
- The *b* determines the **period** of the function
  - The bigger the value of *b* the quicker the function repeats a cycle
- The *d* represents the **principal axis**
  - This is the line that the function fluctuates around

What can be modelled as a sinusoidal model?

- Anything that oscillates (fluctuates periodically)
- Examples include:
  - *D*(*t*) is the depth of water at a shore *t* hours after midnight
  - *T*(*d*) is the temperature of a city *d* days after the 1st January
  - *H*(*t*) is vertical height above ground of a person *t* second after entering a Ferris wheel

What are possible limitations of a sinusoidal model?

- The amplitude is the same for each cycle
  - In real-life this might not be the case
  - The function might get closer to the principal axis over time
- The period is the same for each cycle
  - In real-life this might not be the case
  - The time to complete a cycle might change over time

Exam Tip

- Read and re-read the question carefully, try to get involved in the context of the question!
- **Sketch** a graph of the function being used as the model, use your GDC to help you and focus on the given domain

YOUR NOTES
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Worked Example

The water depth, D, in metres, at a port can be modelled by the function

$$D(t) = 3\sin(15° \times t) + 12, \quad 0 \le t < 24$$

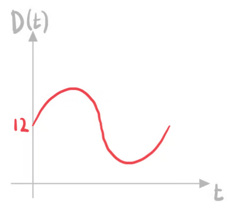
where t is the elapsed time, in hours, since midnight.

a)
Write down the depth of the water at midnight.

Substitute $t = 0$ for midnight

$D(0) = 3\sin(15 \times 0) + 12$

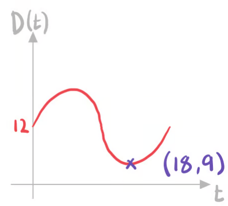
12 m



b)
Find the minimum water depth and the number of hours after midnight that this depth occurs.

Use GDC to find the minimum

Minimum = 9 m
18 hours after midnight



c)
Calculate how long the water depth is at least 13.5 metres each day.

Use GDC to find $D(t) = 13.5$

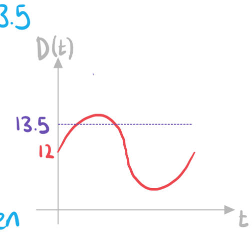
$3\sin(15t) + 12 = 13.5$

$t = 2$ and $t = 10$

Find the difference between the times

$10 - 2 = 8$

8 hours



2.3.6 Strategy for Modelling Functions

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Modelling with Functions

What is a mathematical model?

- A **mathematical model** simplifies a real-world situation so it can be described using mathematics
    - The model can then be used to make predictions
- **Assumptions** about the situation are made in order to simplify the mathematics
- Models can be **refined** (improved) if further information is available or if the model is compared to real-world data

How do I set up the model?

- The question could:
    - give you the equation of the model
    - tell you about the relationship
        - It might say the relationship is linear, quadratic, etc
    - ask you to suggest a **suitable model**
        - Use your knowledge of each model
        - E.g. if it is compound interest then an exponential model is the most appropriate
- You may have to determine a **reasonable domain**
    - Consider real-life context
        - E.g. if dealing with hours in a day then
        - E.g. if dealing with physical quantities (such as length) then
    - Consider the **possible ranges**
        - If the outcome cannot be negative then you want to choose a domain which corresponds to a range with no negative values
        - **Sketching the graph** is helpful to determine a suitable domain

Which models do I need to know?

- Linear
- Piecewise linear
- Quadratic
- Cubic
- Exponential
- Direct variation
- Inverse variation
- Sinusoidal

YOUR NOTES
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Exam Tip

- You need to be familiar with the format of the different types of equations and the general shape of the graphs they produce, you need to always be thinking "does my answer seem appropriate for the given situation?"
- Sketching graphs is key
  - Make sure that you use your GDC to plot the relevant function(s)
  - Sometimes you may have to play around with the zoom function or the axes to make sure that you are focused on the relevant domain

Worked Example

A cliff has a height x metres above the ground. A stone is projected from the edge of the cliff and it travels through the air until it hits the ground and stops. The vertical height, in metres, of the stone above the ground t seconds after being thrown is given by the function:

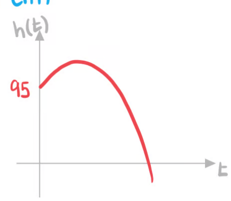
$$h(t) = 95 + 6t - 5t^2.$$

a)
State the value of x.

Initial value is the height of the cliff

$h(0) = 95 + 6(0) - 5(0)^2$

95 m



b)
Determine the domain of $h(t)$.

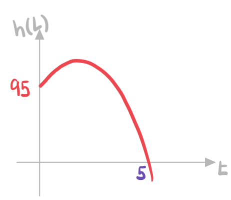
Stone stops at ground when $h(t) = 0$

$95 + 6t - 5t^2 = 0$

$t = 5$ or $t = -3.8$

↑ Reject as time can't be negative

$0 \leq t \leq 5$



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Finding Parameters

What do I do if some of the parameters are unknown?

- For some models you can use your knowledge to find unknown parameters
  - For a linear model $f(x) = mx + c$ you can calculate the *m* by finding the gradient
  - For an exponential model $f(x) = ka^x + c$ you can calculate the *a* by finding the percentage change
  - For a sinusoidal model $f(x) = a\sin(bx) + d$ you can calculate *a* by finding the amplitude
- A general method is to form equations by substituting in given values
  - You can form multiple equations and solve them **simultaneously using your GDC**
    - You could be expected to solve a system of **three simultaneous equations** of three unknowns
  - This method works for all models
- The **initial value** is the value of the function when the variable is 0
  - This is normally one of the parameters in the equation of the model

Exam Tip

- Make sure that any sketches you are asked to make are fully labelled with the coordinates of any important points, e.g. intersections with the axes or other lines, local maxima/minima etc.

YOUR NOTES
⬇

Worked Example

The temperature, T °C, of a cup of coffee is monitored. Initially the temperature is 80°C and 5 minutes later it is 40°C. It is suggested that the temperature follows the model:

$$T(t) = ka^{-t} + 16, \quad t \geq 0$$

where t is the time, in minutes, after the coffee has been made.

a)
State the value of k.

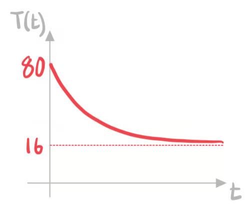
Initially temperature is 80°C

$T(0) = 80$

$ka^{-0} + 16 = 80$

$k + 16 = 80$

$k = 64$



b)
Find the value of a.

After 5 minutes the temperature is 40°C

$T(5) = 40$

$64a^{-5} + 16 = 40$

Solve using GDC

$a = 1.21672...$

$a = 1.22$ (3sf)