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# IB Maths DP

## 2. Functions

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## 2.1 Linear Functions & Graphs

### 2.1.1 Equations of a Straight Line

## Equations of a Straight Line

### How do I find the gradient of a straight line?

- Find two points that the line passes through with coordinates ($x_1$, $y_1$) and ($x_2$, $y_2$)
- The gradient between these two points is calculated by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

  - This is given in the **formula booklet**
- The gradient of a straight line measures its **slope**
  - A line with gradient 1 will go up 1 unit for every unit it goes to the right
  - A line with gradient -2 will go down two units for every unit it goes to the right

### What are the equations of a straight line?

- $y = mx + c$
  - This is the **gradient-intercept form**
  - It clearly shows the gradient *m* and the *y*-intercept (0, c)
- $y - y_1 = m(x - x_1)$
  - This is the **point-gradient form**
  - It clearly shows the gradient *m* and a point on the line ($x_1$, $y_1$)
- $ax + by + d = 0$
  - This is the **general form**
  - You can quickly get the *x*-intercept $\left(-\frac{d}{a}, 0\right)$ and *y*-intercept $\left(0, -\frac{d}{b}\right)$

### How do I find an equation of a straight line?

- You will need the gradient
  - If you are given two points then first find the gradient
- It is easiest to start with the **point-gradient form**
  - then rearrange into whatever form is required
    - multiplying both sides by any denominators will get rid of fractions
- You can check your answer by using your GDC
  - Graph your answer and check it goes through the point(s)
  - If you have two points then you can enter these in the **statistics mode** and find the regression line $y = ax + b$

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## Exam Tip

- A sketch of the graph of the straight line(s) can be helpful, even if not demanded by the question
  - Use your GDC to plot them
- Ensure you state equations of straight lines in the format required
  - Usually $y = mx + c$ or $ax + by + d = 0$
  - Check whether coefficients need to be integers (they usually are for $ax + by + d = 0$)

## Worked Example

The line $l$ passes through the points $(-2, 5)$ and $(6, -7)$.

Find the equation of $l$, giving your answer in the form $ax + by + d = 0$ where $a, b$ and $c$ are integers to be found.

Find the gradient between (-2,5) and (6,-7)

Formula booklet

$m = \frac{-7 - 5}{6 - -2} = -\frac{3}{2}$

| | |
|---|---|
| Gradient formula | $m = \frac{y_2 - y_1}{x_2 - x_1}$ |

Use the point-gradient formula

Formula booklet

| | |
|---|---|
| Equations of a straight line | $y - y_1 = m(x - x_1)$ |

$(x_1, y_1) = (-2, 5) \quad m = -\frac{3}{2}$

$y - 5 = -\frac{3}{2}(x - -2)$ Simplify

$y - 5 = -\frac{3}{2}(x + 2)$

Multiply by denominator

$2(y - 5) = -3(x + 2)$

Expand

$2y - 10 = -3x - 6$

Rearrange

$3x + 2y - 4 = 0$

## Parallel Lines

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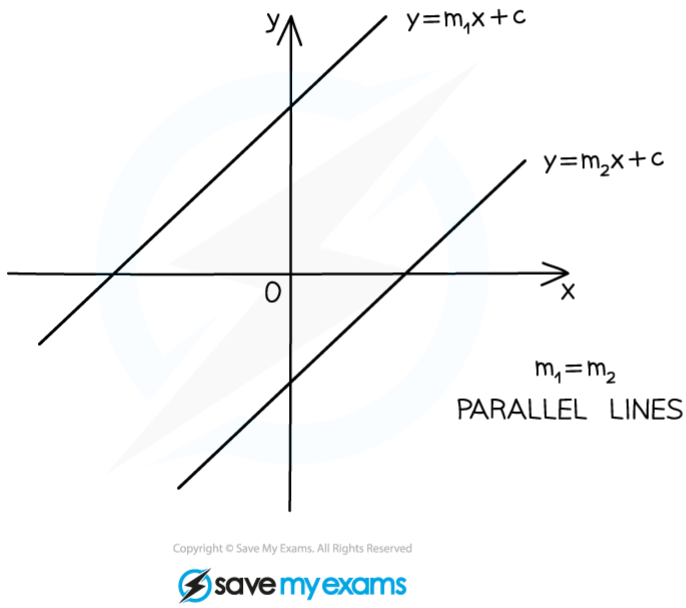
### How are the equations of parallel lines connected?

- **Parallel lines** are always equidistant meaning they never intersect
- Parallel lines have the same gradient
  - If the gradient of line $l_1$ is $m_1$ and gradient of line $l_2$ is $m_2$ then...
    - $m_1 = m_2 \Rightarrow l_1$ & $l_2$ are parallel
    - $l_1$ & $l_2$ are parallel $\Rightarrow m_1 = m_2$
- To determine if two lines are parallel:
  - Rearrange into the gradient-intercept form $y = mx + c$
  - Compare the coefficients of $x$
  - If they are equal then the lines are parallel



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## Worked Example

The line $l$ passes through the point $(4, -1)$ and is parallel to the line with equation $2x - 5y = 3$.

Find the equation of $l$, giving your answer in the form $y = mx + c$.

Rearrange into $y = mx + c$ to find the gradient

$5y = 2x - 3 \Rightarrow y = \frac{2}{5}x - \frac{3}{5}$ $\therefore$ gradient $= \frac{2}{5}$

Parallel lines $\Rightarrow m_1 = m_2$

$m = \frac{2}{5}$

Use the point-gradient formula

Formula booklet

| | |
|---|---|
| Equations of a straight line | $y - y_1 = m(x - x_1)$ |

$(x_1, y_1) = (4, -1)$ $m = \frac{2}{5}$

$y + 1 = \frac{2}{5}(x - 4)$

$y + 1 = \frac{2}{5}x - \frac{8}{5}$

$y = \frac{2}{5}x - \frac{13}{5}$

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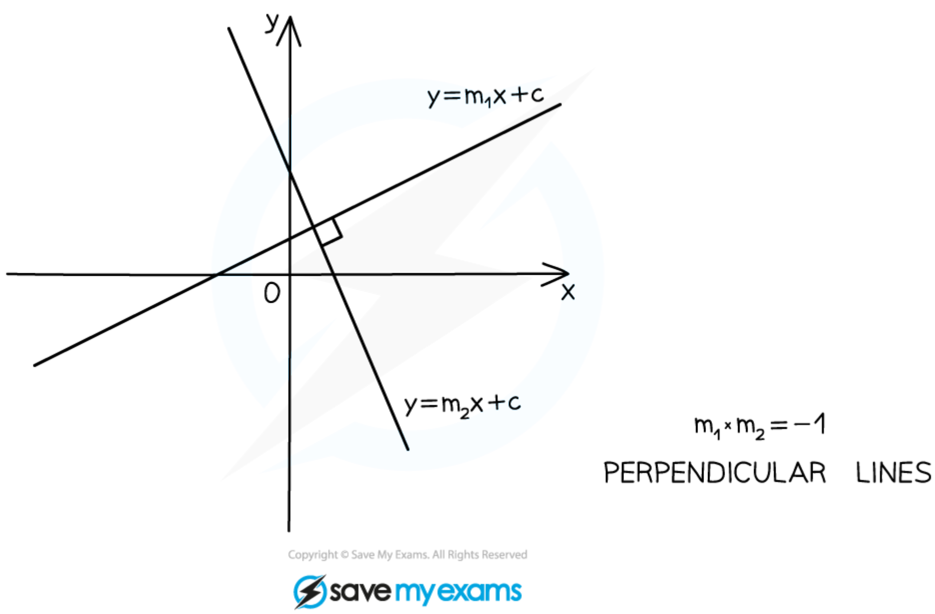
## Perpendicular Lines

### How are the equations of perpendicular lines connected?

- **Perpendicular lines** intersect at right angles
- The gradients of two perpendicular lines are negative reciprocals
  - If the gradient of line $l_1$ is $m_1$ and gradient of line $l_2$ is $m_2$ then...
    - $m_1 \times m_2 = -1 \Rightarrow l_1$ & $l_2$ are perpendicular
    - $l_1$ & $l_2$ are perpendicular $\Rightarrow m_1 \times m_2 = -1$
- To determine if two lines are perpendicular:
  - Rearrange into the gradient-intercept form $y = mx + c$
  - Compare the coefficients of $x$
  - If their product is -1 then they are perpendicular
- Be careful with horizontal and vertical lines
  - $x = p$ and $y = q$ are perpendicular where *p* and *q* are constants



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## Worked Example

The line $l_1$ is given by the equation $3x - 5y = 7$.

The line $l_2$ is given by the equation $y = \frac{1}{4} - \frac{5}{3}x$.

Determine whether $l_1$ and $l_2$ are perpendicular. Give a reason for your answer.

Rearrange $l_1$ into $y = mx + c$ form

$5y = 3x - 7 \Rightarrow y = \frac{3}{5}x - \frac{7}{5}$

Identify gradients

$m_1 = \frac{3}{5}$ $m_2 = -\frac{5}{3}$

$m_1 \times m_2 = -1 \Rightarrow$ Perpendicular lines

$\frac{3}{5} \times -\frac{5}{3} = -1$

**$l_1$ and $l_2$ are perpendicular as $m_1 \times m_2 = -1$**

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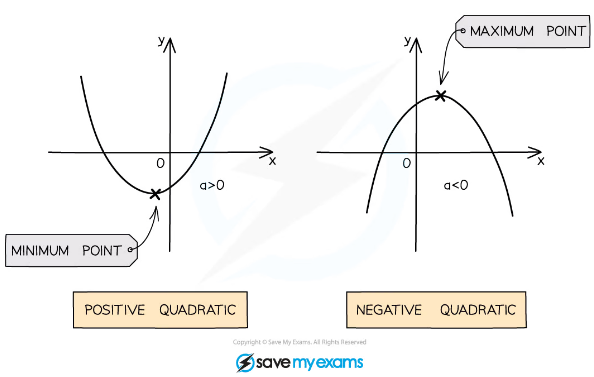
# 2.2 Quadratic Functions & Graphs

## 2.2.1 Quadratic Functions

### Quadratic Functions & Graphs

### What are the key features of quadratic graphs?

- A **quadratic** graph can be written in the form $y = ax^2 + bx + c$ where $a \neq 0$
- The value of *a* affects the shape of the curve
  - If *a* is **positive** the shape is **concave up** ∪
  - If *a* is **negative** the shape is **concave down** ∩
- The ***y*-intercept** is at the point (0, c)
- The **zeros or roots** are the solutions to $ax^2 + bx + c = 0$
  - These can be found by
    - Factorising
    - Quadratic formula
    - Using your GDC
  - These are also called the *x*-intercepts
  - There can be 0, 1 or 2 *x*-intercepts
    - This is determined by the value of the **discriminant**
- There is an **axis of symmetry** at $x = -\dfrac{b}{2a}$
  - This is given in your **formula booklet**
  - If there are two *x*-intercepts then the axis of symmetry goes through the midpoint of them
- The **vertex** lies on the axis of symmetry
  - It can be found by **completing the square**
  - The *x*-coordinate is $x = -\dfrac{b}{2a}$
  - The *y*-coordinate can be found using the GDC or by calculating *y* when $x = -\dfrac{b}{2a}$
  - If *a* is **positive** then the vertex is the **minimum point**
  - If *a* is **negative** then the vertex is the **maximum point**



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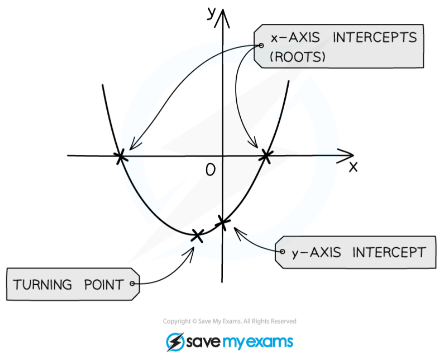


## What are the equations of a quadratic function?

- $f(x) = ax^2 + bx + c$
  - This is the **general form**
  - It clearly shows the *y*-intercept (0, c)
  - You can find the axis of symmetry by $x = -\dfrac{b}{2a}$
    - This is given in the formula booklet
- $f(x) = a(x - p)(x - q)$
  - This is the **factorised form**
  - It clearly shows the roots (*p*, 0) & (*q*, 0)
  - You can find the axis of symmetry by $x = \dfrac{p + q}{2}$
- $f(x) = a(x - h)^2 + k$
  - This is the **vertex form**
  - It clearly shows the vertex (*h*, *k*)
  - The axis of symmetry is therefore $x = h$
  - It clearly shows how the function can be transformed from the graph $y = x^2$
    - Vertical stretch by scale factor *a*
    - Translation by vector $\begin{pmatrix} h \\ k \end{pmatrix}$

## How do I find an equation of a quadratic?

- If you have the **roots** *x* = *p* and *x* = *q*...
  - Write in **factorised form** $y = a(x - p)(x - q)$
  - You will need a third point to find the value of *a*
- If you have the **vertex** (*h*, *k*) then...
  - Write in **vertex form** $y = a(x - h)^2 + k$
  - You will need a second point to find the value of *a*
- If you have **three random points** $(x_1, y_1)$, $(x_2, y_2)$ & $(x_3, y_3)$ then...
  - Write in the **general form** $y = ax^2 + bx + c$
  - Substitute the three points into the equation
  - Form and solve a system of three linear equations to find the values of *a*, *b* & c

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### Exam Tip

- Use your GDC to find the roots and the turning point of a quadratic function
  - You do not need to factorise or complete the square
  - It is good exam technique to sketch the graph from your GDC as part of your working

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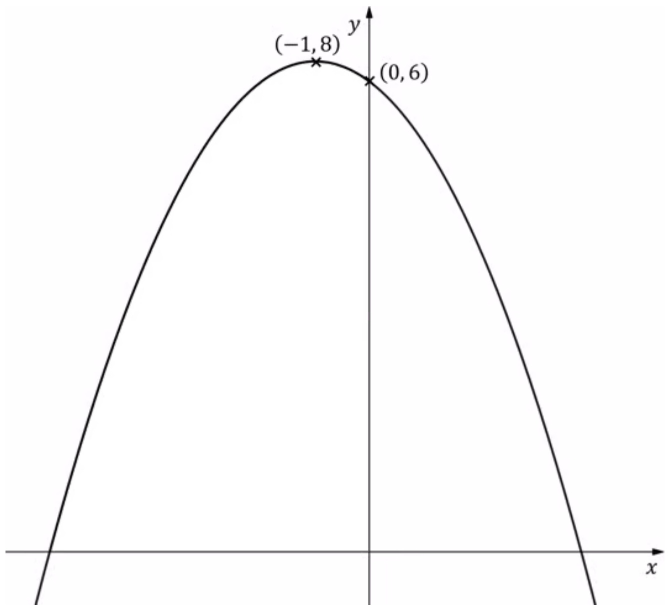
## Worked Example

The diagram below shows the graph of $y = f(x)$, where $f(x)$ is a quadratic function.

The intercept with the $y$-axis and the vertex have been labelled.




Write down an expression for $y = f(x)$.

We have the vertex so use $y = a(x-h)^2 + k$

Vertex $(-1, 8)$ : $y = a(x-(-1))^2 + 8$

$y = a(x+1)^2 + 8$

Substitute the second point

$x = 0, y = 6$ : $6 = a(0+1)^2 + 8$

$6 = a + 8$

$a = -2$

$y = -2(x+1)^2 + 8$

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## 2.2.2 Factorising & Completing the Square

# Factorising Quadratics

### Why is factorising quadratics useful?

- Factorising gives **roots (zeroes or solutions)** of a quadratic
- It gives the ***x*-intercepts** when drawing the graph

### How do I factorise a monic quadratic of the form $x^2 + bx + c$?

- You might be able to spot the factors by **inspection**
  - Especially if c is a **prime number**
- Otherwise find two numbers *m* and *n* ..
  - A sum equal to *b*
    - $m + n = b$
  - A product equal to c
    - $mn = c$
- Rewrite $bx$ as $mx + nx$
- Use this to factorise $x^2 + mx + nx + c$
- A shortcut is to write:
  - $(x + m)(x + n)$

### How do I factorise a non-monic quadratic of the form $ax^2 + bx + c$?

- If *a*, *b* & *c* have a common factor then first factorise that out to leave a quadratic with coefficients that have **no common factors**
- You might be able to spot the factors by **inspection**
  - Especially if *a* and/or c are **prime numbers**
- Otherwise find two numbers *m* and *n* ..
  - A sum equal to *b*
    - $m + n = b$
  - A product equal to *ac*
    - $mn = ac$
- Rewrite $bx$ as $mx + nx$
- Use this to factorise $ax^2 + mx + nx + c$
- A shortcut is to write:
  - $\dfrac{(ax + m)(ax + n)}{a}$
  - Then factorise common factors from numerator to cancel with the *a* on the denominator

### How do I use the difference of two squares to factorise a quadratic of the form $A^2x^2 - C^2$?

- The **difference of two squares** can be used when...
  - There is **no *x* term**
  - The **constant term is a negative**
- Square root the two terms $A^2x^2$ & $C^2$
- The two factors are the **sum of square roots** and the **difference of the square roots**
- A shortcut is to write:

- $(Ax + C)(Ax - C)$

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### Exam Tip

- You can deduce the factors of a quadratic function by using your GDC to find the solutions of a quadratic equation
  - Using your GDC, the quadratic equation $6x^2 + x - 2 = 0$ has solutions $x = -\frac{2}{3}$ and $x = \frac{1}{2}$
  - Therefore the factors would be $(3x + 2)$ and $(2x - 1)$
  - i.e. $6x^2 + x - 2 = (3x + 2)(2x - 1)$

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## Worked Example

Factorise fully:

a)
$x^2 - 7x + 12$.

Find two numbers m and n such that

$m+n = b = -7$ $\quad$ $mn = c = 12$

$-4 + -3 = -7$ $\quad$ $-4 \times -3 = 12$

Split $-7x$ up and factorise

$x^2 - 4x - 3x + 12$

$x(x-4) - 3(x-4)$

$(x-3)(x-4)$

Shortcut

$(x+m)(x+n)$

$(x-3)(x-4)$

b)
$4x^2 + 4x - 15$.

Find two numbers m and n such that

$m+n = b = 4$ $\quad$ $mn = ac = 4 \times -15 = -60$

$10 + -6 = 4$ $\quad$ $10 \times -6 = -60$

Split $4x$ up and factorise

$4x^2 + 10x - 6x - 15$

$2x(2x+5) - 3(2x+5)$

$(2x-3)(2x+5)$

Shortcut

$\frac{(ax+m)(ax+n)}{a}$

$\frac{(4x+10)(4x-6)}{4}$

$\frac{2(2x+5)\,2(2x-3)}{4}$

$(2x-3)(2x+5)$

c)
$18 - 50x^2$.

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Factorise the common factor

$2(9-25x^2)$

Use difference of two squares

$2(3-5x)(3+5x)$

## Completing the Square

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### Why is completing the square for quadratics useful?

- Completing the square gives the **maximum/minimum** of a quadratic function
  - This can be used to define the **range** of the function
- It gives the **vertex** when drawing the graph
- It can be used to **solve quadratic equations**
- It can be used to derive the **quadratic formula**

### How do I complete the square for a monic quadratic of the form $x^2 + bx + c$?

- **Half the value of *b*** and write $\left(x + \frac{b}{2}\right)^2$
  - This is because $\left(x + \frac{b}{2}\right)^2 = x^2 + bx + \frac{b^2}{4}$
- **Subtract the unwanted** $\frac{b^2}{4}$ **term** and **add on the constant c**
  - $\left(x + \frac{b}{2}\right)^2 - \frac{b^2}{4} + c$

### How do I complete the square for a non-monic quadratic of the form $ax^2 + bx + c$?

- **Factorise out the *a*** from the **terms involving *x***
  - $a\left(x^2 + \frac{b}{a}x\right) + x$
  - Leaving the c alone will **avoid working with lots of fractions**
- **Complete the square** on the quadratic term
  - **Half** $\frac{b}{a}$ and write $\left(x + \frac{b}{2a}\right)^2$
    - This is because $\left(x + \frac{b}{2a}\right)^2 = x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}$
  - **Subtract the unwanted** $\frac{b^2}{4a^2}$ **term**
- **Multiply by *a*** and **add the constant c**
  - $a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}\right] + c$
  - $a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$

**Exam Tip**

- Some questions may not use the phrase "completing the square" so ensure you can recognise a quadratic expression or equation written in this form
  - $a(x - h)^2 + k\ (= 0)$

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## Worked Example

Complete the square:

a)
$x^2 - 8x + 3$.

Half b and subtract its square

$(x-4)^2 - 4^2 + 3$

$(x-4)^2 - 13$

b)
$3x^2 + 12x - 5$.

Factorise the 3 from the x terms

$3(x^2 + 4x) - 5$

Complete the square on $x^2 + 4x$

$3\left((x+2)^2 - 2^2\right) - 5$

Simplify

$3\left((x+2)^2 - 4\right) - 5$

$3(x+2)^2 - 12 - 5$

$3(x+2)^2 - 17$

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## Solving Quadratics Equations

### How do I decide the best method to solve a quadratic equation?

- A quadratic equation is of the form $ax^2 + bx + c = 0$
- If it is a calculator paper then use your GDC to solve the quadratic
- If it is a non-calculator paper then...
  - you can always use the **quadratic formula**
  - you can **factorise** if it can be factorised with integers
  - you can always **complete the square**

### How do I solve a quadratic equation by the quadratic formula?

- If necessary **rewrite** in the form $ax^2 + bx + c = 0$
- **Clearly identify** the values of *a*, *b* & *c*
- **Substitute** the values into the formula
  - $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
    - This is given in the **formula booklet**
- **Simplify** the solutions as much as possible

### How do I solve a quadratic equation by factorising?

- **Factorise** to rewrite the quadratic equation in the form $a(x-p)(x-q) = 0$
- Set each factor to zero and **solve**
  - $x - p = 0 \Rightarrow x = p$
  - $x - q = 0 \Rightarrow x = q$

### How do I solve a quadratic equation by completing the square?

- **Complete the square** to rewrite the quadratic equation in the form $a(x-h)^2 + k = 0$
- Get the squared term by itself
  - $(x-h)^2 = -\dfrac{k}{a}$
- If $\left(-\dfrac{k}{a}\right)$ is **negative** then there will be **no solutions**
- If $\left(-\dfrac{k}{a}\right)$ is **positive** then there will be **two values** for $x - h$
  - $x - h = \pm\sqrt{-\dfrac{k}{a}}$
- **Solve** for *x*
  - $x = h \pm \sqrt{-\dfrac{k}{a}}$

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### Exam Tip

- When using the quadratic formula with awkward values or fractions you may find it easier to deal with the " $b^2 - 4ac$ " (**discriminant**) first
  - This can help avoid numerical and negative errors, improving accuracy

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## Worked Example

Solve the equations:

a)
$4x^2 + 4x - 15 = 0$.

This can be factorised

$(2x+5)(2x-3) = 0$

$2x+5 = 0$ or $2x-3 = 0$

$x = -\frac{5}{2}$ or $x = \frac{3}{2}$

b)
$3x^2 + 12x - 5 = 0$.

This can not be factorised but $3x^2$ and $12x$ have a common factor so complete the square

$3(x+2)^2 - 17 = 0$

Rearrange

$(x+2)^2 = \frac{17}{3}$

Remember ±

$x + 2 = \pm\sqrt{\frac{17}{3}}$

$x = -2 \pm \sqrt{\frac{17}{3}}$

c)
$7 - 3x - 5x^2 = 0$.

This can not be factorised so use formula

Formula booklet

| Solutions of a quadratic equation | $ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \; a \neq 0$ |
|---|---|

$a = -5 \quad b = -3 \quad c = 7$

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(-5)(7)}}{2(-5)}$$

$$= \frac{3 \pm \sqrt{9 + 140}}{-10}$$

$$x = -\frac{3 \pm \sqrt{149}}{10}$$

2.2.4 Quadratic Inequalities

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## Quadratic Inequalities

### What affects the inequality sign when rearranging a quadratic inequality?

- The inequality sign is **unchanged** by...
  - **Adding/subtracting** a term to both sides
  - **Multiplying/dividing** both sides by a **positive term**
- The inequality sign **flips** (< changes to >) when...
  - **Multiplying/dividing** both sides by a **negative term**

### How do I solve a quadratic inequality?

- **STEP 1**: **Rearrange** the inequality into quadratic form with a **positive squared term**
  - $ax^2 + bx + c > 0$
  - $ax^2 + bx + c \geq 0$
  - $ax^2 + bx + c < 0$
  - $ax^2 + bx + c \leq 0$
- **STEP 2**: Find the **roots** of the quadratic equation
  - Solve $ax^2 + bx + c = 0$ to get $x_1$ and $x_2$ where $x_1 < x_2$
- **STEP 3**: **Sketch** a graph of the quadratic and label the roots
  - As the squared term is positive it will be **concave up** so "U" shaped
- **STEP 4**: **Identify** the **region** that satisfies the inequality
  - If you want the graph to be **above the *x*-axis** then choose the region to be the **two intervals outside** of the two roots
  - If you want the graph to be **below the *x*-axis** then choose the region to be the **interval between** the two roots
  - For $ax^2 + bx + c > 0$
    - The solution is $x < x_1$ **or** $x > x_2$
  - For $ax^2 + bx + c \geq 0$
    - The solution is $x \leq x_1$ **or** $x \geq x_2$
  - For $ax^2 + bx + c < 0$
    - The solution is $x_1 < x < x_2$
  - For $ax^2 + bx + c \leq 0$
    - The solution is $x_1 \leq x \leq x_2$

### How do I solve a quadratic inequality of the form $(x - h)^2 < n$ or $(x - h)^2 > n$?

- The safest way is by following the steps above
  - Expand and rearrange
- A **common mistake** is writing $x - h < \pm\sqrt{n}$ or $x - h > \pm\sqrt{n}$
  - This is **NOT correct**!
- The correct solution to $(x - h)^2 < n$ is
  - $|x - h| < \sqrt{n}$ which can be written as $-\sqrt{n} < x - h < \sqrt{n}$
  - The **final solution** is $h - \sqrt{n} < x < h + \sqrt{n}$
- The correct solution to $(x - h)^2 > n$ is
  - $|x - h| > \sqrt{n}$ which can be written as $x - h < -\sqrt{n}$ or $x - h > \sqrt{n}$

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- The **final solution** is $x < h - \sqrt{n}$ or $x > h + \sqrt{n}$

### Exam Tip

- It is easiest to sketch the graph of a quadratic when it has a positive $x^2$ term, so rearrange first if necessary
- Use your GDC to help select the correct region(s) for the inequality
- Some makes/models of GDC may have the ability to solve inequalities directly
  - However unconventional notation may be used to display the answer (e.g. $6 > x > 3$ rather than $3 < x < 6$)
  - The safest method is to **always** sketch the graph

### Worked Example

Find the set of values which satisfy $3x^2 + 2x - 6 > x^2 + 4x - 2$.

Step 1 : Rearrange

$(3x^2 + 2x - 6) - (x^2 + 4x - 2) > 0$ — This way gives $a > 0$

$2x^2 - 2x - 4 > 0$

$x^2 - x - 2 > 0$ — Divide by factor of 2

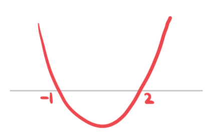
Step 2 : Find the roots

$x^2 - x - 2 = 0$

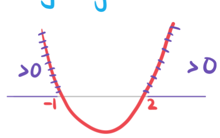
$(x - 2)(x + 1) = 0$

$x = 2$ or $x = -1$

Step 3 : Sketch




Step 4 : Identify region




$x < -1$ or $x > 2$

2.2.5 Discriminants

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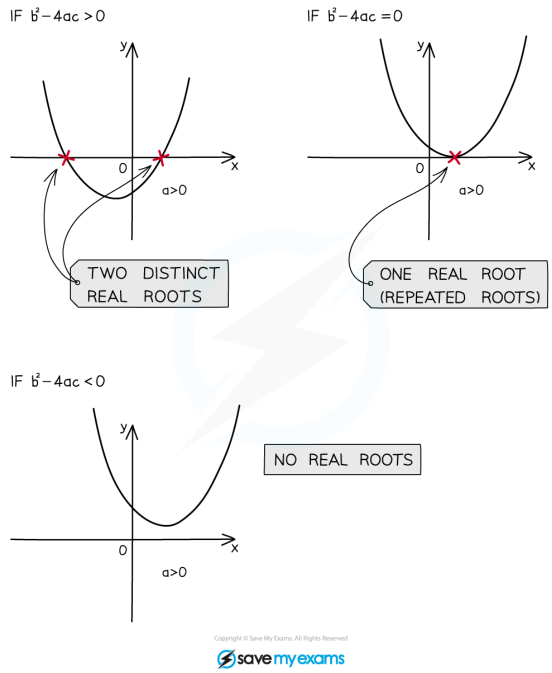
## Discriminants

### What is the discriminant of a quadratic function?

- The discriminant of a quadratic is denoted by the Greek letter Δ (upper case delta)
- For the quadratic function the discriminant is given by
  - $\Delta = b^2 - 4ac$
    - This is given in the **formula booklet**
- The discriminant is the expression that is square rooted in the **quadratic formula**

### How does the discriminant of a quadratic function affect its graph and roots?

- If Δ > 0 then $\sqrt{b^2 - 4ac}$ and $-\sqrt{b^2 - 4ac}$ are **two distinct values**
  - The equation $ax^2 + bx + c = 0$ has **two distinct real solutions**
  - The graph of $y = ax^2 + bx + c$ has **two distinct real roots**
    - This means the graph **crosses** the *x*-axis **twice**
- If Δ = 0 then $\sqrt{b^2 - 4ac}$ and $-\sqrt{b^2 - 4ac}$ are **both zero**
  - The equation $ax^2 + bx + c = 0$ has **one repeated real solution**
  - The graph of $y = ax^2 + bx + c$ has **one repeated real root**
    - This means the graph **touches** the *x*-axis at **exactly one point**
    - This means that the **x-axis** is a **tangent** to the graph
- If Δ < 0 then $\sqrt{b^2 - 4ac}$ and $-\sqrt{b^2 - 4ac}$ are **both undefined**
  - The equation $ax^2 + bx + c = 0$ has **no real solutions**
  - The graph of $y = ax^2 + bx + c$ has **no real roots**
    - This means the graph **never touches** the ***x*-axis**
    - This means that graph is **wholly above** (or **below**) the ***x*-axis**




## Forming equations and inequalities using the discriminant

- Often at least one of the coefficients of a quadratic is **unknown**
  - Questions usually use the letter *k* for the unknown constant
- You will be given a fact about the quadratic such as:
  - The **number of solutions** of the equation
  - The **number of roots** of the graph
- To find the **value or range of values** of *k*
  - Find an **expression for the discriminant**
    - Use $\Delta = b^2 - 4ac$
  - Decide whether $\Delta > 0$, $\Delta = 0$ or $\Delta < 0$
    - If the question says there are **real roots** but does not specify how many then use $\Delta \geq 0$
  - **Solve** the resulting equation or inequality

### Exam Tip

- Questions will rarely use the word discriminant so it is important to recognise when its use is required
  - Look for
    - a number of roots or solutions being stated
    - whether and/or how often the graph of a quadratic function intercepts the $x$-axis
- Be careful setting up inequalities that concern "two real roots" ( $\Delta \geq 0$ ) as opposed to "two real distinct roots" ( $\Delta > 0$ )

YOUR NOTES

## Worked Example

A function is given by $f(x) = 2kx^2 + kx - k + 2$, where $k$ is a constant. The graph of $y = f(x)$ has two distinct real roots.

a)
Show that $9k^2 - 16k > 0$.

Two distinct real roots $\Rightarrow \Delta > 0$

Formula booklet

| | |
|---|---|
| Discriminant | $\Delta = b^2 - 4ac$ |

$a = 2k \quad b = k \quad c = (-k+2)$

$\Delta = k^2 - 4(2k)(-k+2)$

$= k^2 + 8k^2 - 16k$

$= 9k^2 - 16k$

$\Delta > 0 \Rightarrow 9k^2 - 16k > 0$

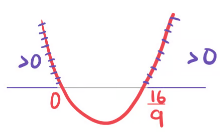
b)
Hence find the set of possible values of $k$.

Solve the inequality

$9k^2 - 16k = 0$

$k(9k - 16) = 0$

$k = 0$ or $k = \frac{16}{9}$




$k < 0$ or $k > \frac{16}{9}$

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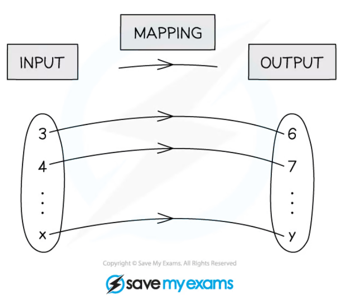
# 2.3 Functions Toolkit

## 2.3.1 Language of Functions

### Language of Functions

### What is a mapping?

- A **mapping transforms** one set of values (**inputs**) into another set of values (**outputs**)
- Mappings can be:
  - **One-to-one**
    - Each input gets mapped to **exactly one unique** output
    - No two inputs are mapped to the same output
    - For example: A mapping that cubes the input
  - **Many-to-one**
    - Each input gets mapped to **exactly one** output
    - Multiple inputs can be mapped to the same output
    - For example: A mapping that squares the input
  - **One-to-many**
    - An input can be mapped to **more than one** output
    - No two inputs are mapped to the same output
    - For example: A mapping that gives the numbers which when squared equal the input
  - **Many-to-many**
    - An input can be mapped to **more than one** output
    - Multiple inputs can be mapped to the same output
    - For example: A mapping that gives the factors of the input




### What is a function?

- A **function** is a mapping between two sets of numbers where **each input** gets mapped to **exactly one output**
  - The output does not need to be unique
- **One-to-one** and **many-to-one** mappings are functions
- A mapping is a function if its graph passes the **vertical line test**
  - Any **vertical line** will intersect with the graph **at most once**

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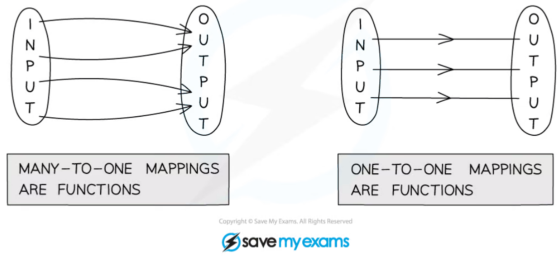


## What notation is used for functions?

- Functions are denoted using letters (such as $f$, $v$, $g$, etc)
  - A function is followed by a variable in a bracket
  - This shows the input for the function
  - The letter $f$ is used most commonly for functions and will be used for the remainder of this revision note
- $f(x)$ represents an expression for the value of the function $f$ when evaluated for the variable $x$
- Function notation gets rid of the need for words which makes it **universal**
  - $f = 5$ when $x = 2$ can simply be written as $f(2) = 5$

## What are the domain and range of a function?

- The **domain** of a function is the set of values that are used as **inputs**
- A domain should be stated with a function
  - If a domain is not stated then it is assumed the domain is all the real values which would work as inputs for the function
  - Domains are expressed in terms of the input
    - $x \leq 2$
- The **range** of a function is the set of values that are given as **outputs**
  - The range depends on the domain
  - Ranges are expressed in terms of the output
    - $f(x) \geq 0$
- To graph a function we use the **inputs as the *x*-coordinates** and the **outputs as the *y*-coordinates**
  - $f(2) = 5$ corresponds to the coordinates (2, 5)
- Graphing the function can help you visualise the range
- Common sets of numbers have special symbols:
  - $\mathbb{R}$ represents all the real numbers that can be placed on a number line
    - $x \in \mathbb{R}$ means $x$ is a real number
  - $\mathbb{Q}$ represents all the rational numbers $\frac{a}{b}$ where *a* and *b* are integers and $b \neq 0$
  - $\mathbb{Z}$ represents all the integers (positive, negative and zero)
    - $\mathbb{Z}^+$ represents positive integers
  - $\mathbb{N}$ represents the natural numbers (0,1,2,3...)

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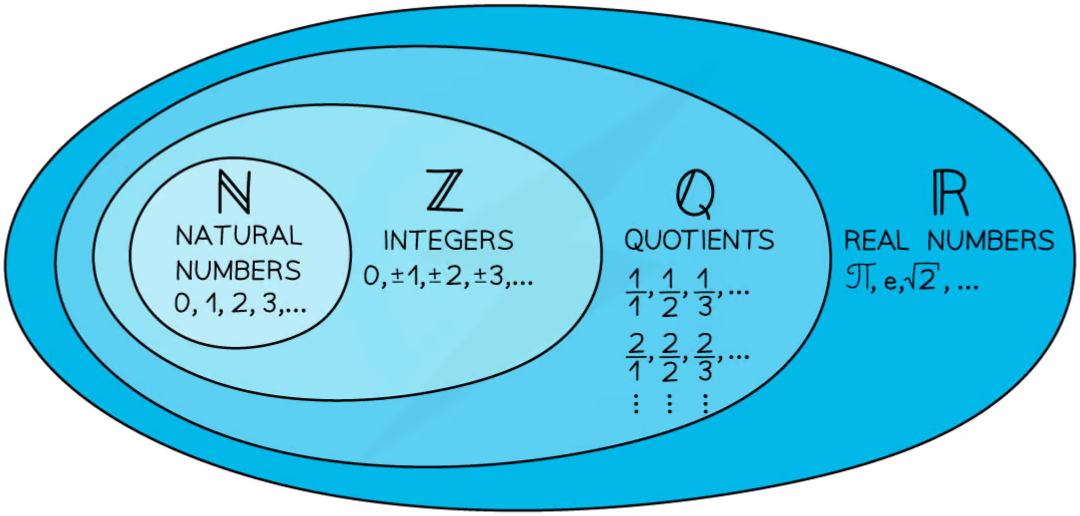


## What are piecewise functions?

- **Piecewise functions** are defined by different functions depending on which interval the input is in
  - E.g. $f(x) = \begin{cases} x+1 & x \le 5 \\ 2x-4 & 5 < x < 10 \\ x^2 & 10 \le x \le 20 \end{cases}$
- The region for the individual functions **cannot overlap**
- To evaluate a piecewise function for a particular value $x = k$
  - Find which interval includes $k$
  - Substitute $x = k$ into the corresponding function
- The function **may or may not be continuous** at the ends of the intervals
  - In the example above the function is
    - continuous at $x = 5$ as $5 + 1 = 2(5) - 4$
    - not continuous at $x = 10$ as $2(10) - 4 \ne 10^2$

### Exam Tip

- Questions may refer to "the largest possible domain"
  - This would usually be $x \in \mathbb{R}$ unless $\mathbb{N}$, $\mathbb{Z}$ or $\mathbb{Q}$ has already been stated
  - There are usualy some exceptions
    - e.g. square roots; $x \ge 0$ for a function involving $\sqrt{x}$
    - e.g. reciprocal functions; $x \ne 2$ for a function with denominator $(x-2)$

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## Worked Example

For the function $f(x) = x^3 + 1,\ 2 \le x \le 10$:

a)
write down the value of $f(7)$.

Substitute $x = 7$

$f(7) = 7^3 + 1$

$f(7) = 344$

b)
find the range of $f(x)$.

Find the values of $x^3+1$ when $2 \le x \le 10$

$2 \le x \le 10$

$8 \le x^3 \le 1000$

$9 \le x^3+1 \le 1001$

$9 \le f(x) \le 1001$

2.3.2 Composite & Inverse Functions

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# Composite Functions

## What is a composite function?

- A **composite function** is where a function is applied to another function
- A composite function can be denoted
  - $(f \circ g)(x)$
  - $fg(x)$
  - $f(g(x))$
- The order matters
  - $(f \circ g)(x)$ means:
    - First apply *g* to *x* to get $g(x)$
    - Then apply *f* to the previous output to get $f(g(x))$
    - Always start with the function **closest to the variable**
  - $(f \circ g)(x)$ is not usually equal to $(g \circ f)(x)$

## How do I find the domain and range of a composite function?

- The domain of $f \circ g$ is the set of values of $x$...
  - which are a **subset** of the **domain of *g***
  - which maps *g* to a value that is in the **domain of *f***
- The range of $f \circ g$ is the set of values of $x$...
  - which are a **subset** of the **range of *f***
  - found by **applying *f*** to the **range of g**
- To find the **domain** and **range** of $f \circ g$
  - First find the **range of *g***
  - **Restrict** these values to the values that are **within the domain** of ***f***
    - The **domain** is the set of values that **produce the restricted range** of *g*
    - The **range** is the set of values that are **produced using the restricted range** of *g* as the domain for *f*
- For example: let $f(x) = 2x + 1,\ -5 \le x \le 5$ and $g(x) = \sqrt{x},\ 1 \le x \le 49$
  - The **range of g** is $1 \le g(x) \le 7$
    - **Restricting** this to fit the **domain of *f*** results in $1 \le g(x) \le 5$
  - The **domain** of $f \circ g$ is therefore $1 \le x \le 25$
    - These are the values of *x* which map to $1 \le g(x) \le 5$
  - The **range** of $f \circ g$ is therefore $3 \le (f \circ g)(x) \le 11$
    - These are the values which *f* maps $1 \le g(x) \le 5$ to

### Exam Tip

- Make sure you know what your GDC is capable of with regard to functions
  - You may be able to store individual functions and find composite functions and their values for particular inputs
  - You may be able to graph composite functions directly and so deduce their domain and range from the graph
- $ff(x)$ is not the same as $[f(x)]^2$

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## Worked Example

Given $f(x)=\sqrt{x+4}$ and $g(x)=3+2x$:

a)
Write down the value of $(g\circ f)(12)$.

First apply function closest to input

$(g\circ f)(12) = g(f(12))$

$f(12) = \sqrt{12+4} = \sqrt{16} = 4$

$g(4) = 3 + 2(4) = 11$

$(g\circ f)(12) = 11$

b)
Write down an expression for $(f\circ g)(x)$.

First apply function closest to input

$(f\circ g)(x) = f(g(x))$

$= f(3+2x)$

$= \sqrt{3+2x+4}$

$(f\circ g)(x) = \sqrt{7+2x}$

c)
Write down an expression for $(g\circ g)(x)$.

$(g\circ g)(x) = g(g(x))$

$= g(3+2x)$

$= 3 + 2(3+2x)$

$= 3 + 6 + 4x$

$(g\circ g)(x) = 9 + 4x$

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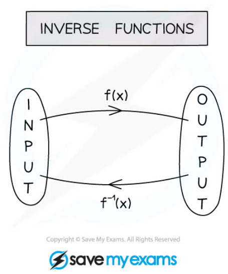
## Inverse Functions

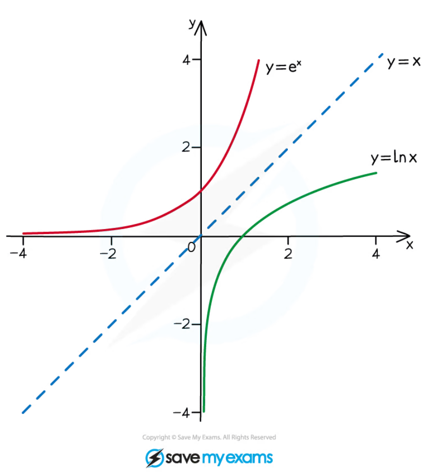
### What is an inverse function?

- **Only one-to-one** functions have inverses
- A function has an inverse if its graph passes the **horizontal line test**
  - Any **horizontal line** will intersect with the graph **at most once**
- The **identity function** $\text{id}$ maps each value to itself
  - $\text{id}(x) = x$
- If $f \circ g$ and $g \circ f$ have the **same effect as the identity function** then $f$ and $g$ are **inverses**
- Given a function $f(x)$ we denote the **inverse function** as $f^{-1}(x)$
- An inverse function **reverses the effect** of a function
  - $f(2) = 5$ means $f^{-1}(5) = 2$
- Inverse functions are used to solve equations
  - The solution of $f(x) = 5$ is $x = f^{-1}(5)$
- A composite function made of $f$ and $f^{-1}$ has the **same effect as the identity function**
  - $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$




### What are the connections between a function and its inverse function?

- The **domain of a function** becomes the **range of its inverse**
- The **range of a function** becomes the **domain of its inverse**
- The graph of $y = f^{-1}(x)$ is a **reflection** of the graph $y = f(x)$ in the line $y = x$
  - Therefore solutions to $f(x) = x$ or $f^{-1}(x) = x$ will also be solutions to $f(x) = f^{-1}(x)$
    - There could be other solutions to $f(x) = f^{-1}(x)$ that don't lie on the line $y = x$




### How do I find the inverse of a function?

- STEP 1: **Swap** the $x$ and $y$ in $y = f(x)$
  - If $y = f^{-1}(x)$ then $x = f(y)$
- STEP 2: **Rearrange** $x = f(y)$ to make $y$ the subject
- Note this can be done in any order
  - Rearrange $y = f(x)$ to make $x$ the subject
  - Swap $x$ and $y$

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### Exam Tip

- Remember that an inverse function is a reflection of the original function in the line $y = x$
  - Use your GDC to plot the function and its inverse on the same graph to visually check this
- $f^{-1}(x)$ is not the same as $\frac{1}{f(x)}$

## Worked Example

For the function $f(x) = \dfrac{2x}{x-1}$, $x > 1$:

a)
Find the inverse of $f(x)$.

Let $y = f^{-1}(x)$ and rearrange $x = f(y)$

$x = \dfrac{2y}{y-1}$

$x(y-1) = 2y$

$xy - x = 2y$

$xy - 2y = x$

$y(x-2) = x$

$y = \dfrac{x}{x-2}$

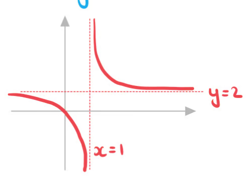
$f^{-1}(x) = \dfrac{x}{x-2}$

b)
Find the domain of $f^{-1}(x)$.

Domain of $f^{-1}$ is the range of $f$

Sketch $y = f(x)$ to see range

For $x > 1$, $f(x) > 2$




Domain of $f^{-1}$ : $x > 2$

c)
Find the value of $k$ such that $f(k) = 6$.

Use inverse $f(a) = b \iff a = f^{-1}(b)$

$k = f^{-1}(6) = \dfrac{6}{6-2}$

$k = \dfrac{3}{2}$

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2.3.3 Graphing Functions

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## Graphing Functions

### How do I graph the function *y* = *f*(*x*)?

- A point $(a,\ b)$ lies on the graph $y = f(x)$ if $f(a) = b$
- The **horizontal axis** is used for the **domain**
- The **vertical axis** is used for the **range**
- You will be able to graph some functions by hand
- For some functions you will need to use your GDC
- You might be asked to graph the **sum** or **difference** of two functions
    - Use your GDC to graph $y = f(x) + g(x)$ or $y = f(x) - g(x)$
    - Just type the functions into the graphing mode

### What is the difference between "draw" and "sketch"?

- If asked to sketch you should:
    - Show the general shape
    - Label any key points such as the intersections with the axes
    - Label the axes
- If asked to draw you should:
    - Use a pencil and ruler
    - Draw to scale
    - Plot any points **accurately**
    - Join points with a straight line or smooth curve
    - Label any key points such as the intersections with the axes
    - Label the axes

### How can my GDC help me sketch/draw a graph?

- You use your GDC to plot the graph
    - Check the scales on the graph to make sure you see the full shape
- Use your GDC to find any key points
- Use your GDC to check specific points to help you plot the graph

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# Key Features of Graphs

## What are the key features of graphs?

- You should be familiar with the following key features and know how to use your GDC to find them
- Local minimums/maximums
    - These are points where the graph has a minimum/maximum for a small region
    - They are also called **turning points**
        - This is where the graph changes its direction between upwards and downwards directions
    - A graph can have multiple local minimums/maximums
    - A local minimum/maximum is not necessarily the minimum/maximum of the whole graph
        - This would be called the **global** minimum/maximum
    - For quadratic graphs the minimum/maximum is called the **vertex**
- Intercepts
    - $y$ – intercepts are where the graph crosses the $y$-axis
        - At these points $x = 0$
    - $x$ – intercepts are where the graph crosses the $x$-axis
        - At these points $y = 0$
        - These points are also called the **zeros of the function** or **roots of the equation**
- Symmetry
    - Some graphs have lines of symmetry
        - A quadratic will have a vertical line of symmetry
- Asymptotes
    - These are lines which the graph will get closer to but not cross
    - These can be horizontal or vertical
        - Exponential graphs have horizontal asymptotes
        - Graphs of variables which vary inversely can have vertical and horizontal asymptotes

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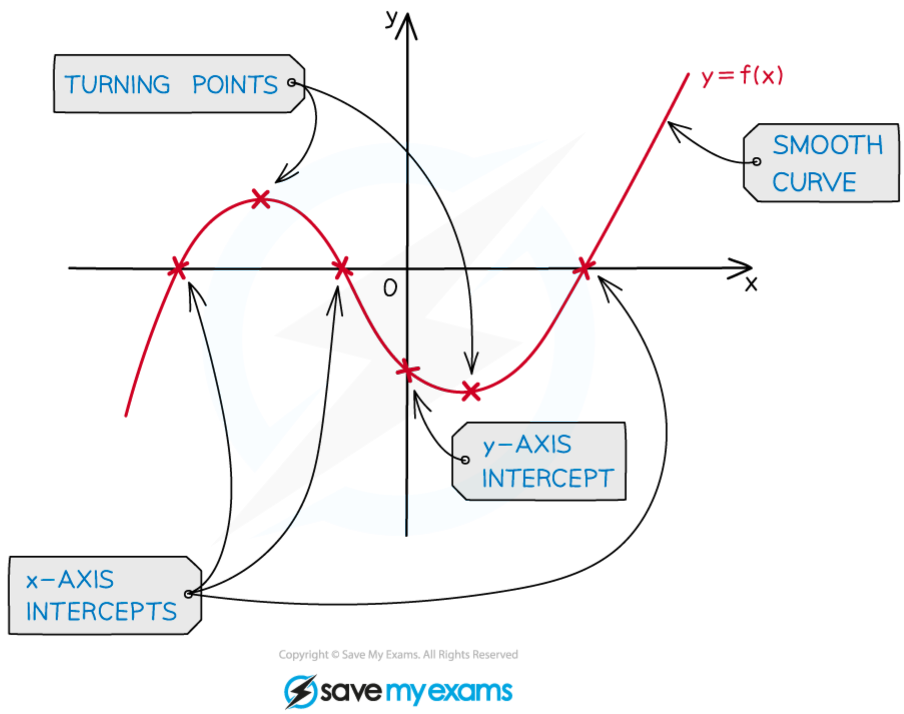


### Exam Tip

- Most GDC makes/models will not plot/show asymptotes just from inputting a function
  - Add the asymptotes as additional graphs for your GDC to plot
  - You can then check the equations of your asymptotes visually
  - You may have to zoom in or change the viewing window options to confirm an asymptote
- Even if using your GDC to plot graphs and solve problems sketching them as part of your working is good exam technique
  - Label the key features of the graph and anything else relevant to the question on your sketch

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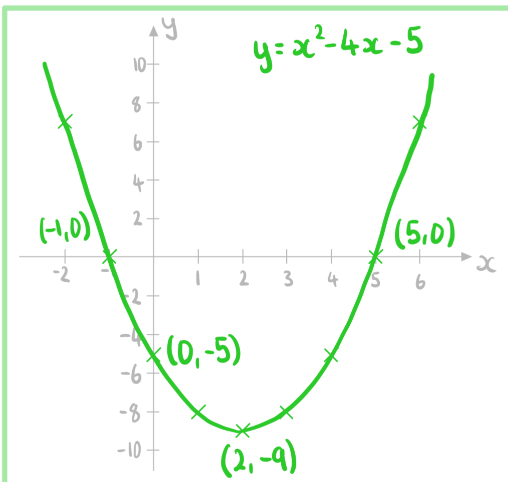
## Worked Example

Two functions are defined by

$$f(x) = x^2 - 4x - 5 \text{ and } g(x) = 2 + \frac{1}{x+1}.$$

a)
Draw the graph $y = f(x)$.

Draw means accurately
Use GDC to find vertex, roots and y-intercepts
Vertex = (2, -9)
Roots = (-1, 0) and (5, 0)
y-intercept = (0, -5)




b)
Sketch the graph $y = g(x)$.

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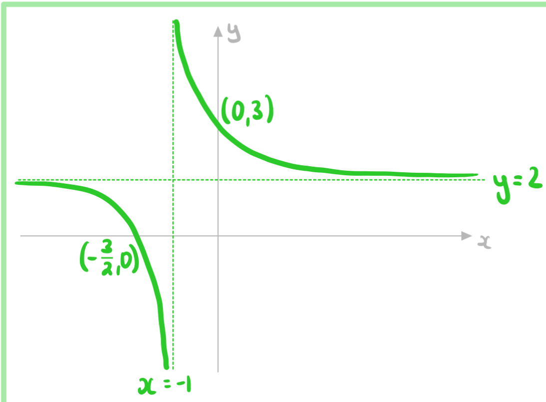
Sketch means rough but showing key points

Use GDC to find $x$ and $y$-intercepts and asymptotes

$x$-intercept $= \left(-\frac{3}{2}, 0\right)$

$y$-intercept $= (0, 3)$

Asymptotes : $x = -1$ and $y = 2$



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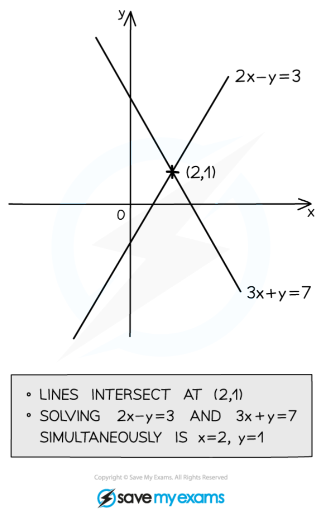
# Intersecting Graphs

## How do I find where two graphs intersect?

- Plot both graphs on your GDC
- Use the intersect function to find the intersections
- Check if there is more than one point of intersection




## How can I use graphs to solve equations?

- One method to solve equations is to use graphs
- To solve $f(x) = a$
  - Plot the two graphs $y = f(x)$ and $y = a$ on your GDC
  - Find the points of intersections
  - The ***x*-coordinates** are the **solutions** of the equation
- To solve $f(x) = g(x)$
  - Plot the two graphs $y = f(x)$ and $y = g(x)$ on your GDC
  - Find the points of intersections
  - The ***x*-coordinates** are the **solutions** of the equation
- Using graphs makes it easier to see **how many solutions** an equation will have

### Exam Tip

- You can use graphs to solve equations
  - Questions will not necessarily ask for a drawing/sketch or make reference to graphs
  - Use your GDC to plot the equations and find the intersections between the graphs

YOUR NOTES

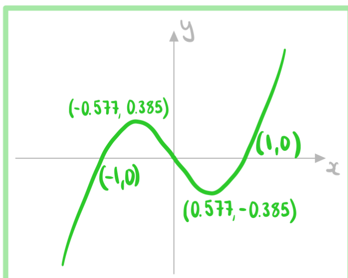
## Worked Example

Two functions are defined by

$$f(x) = x^3 - x \text{ and } g(x) = \frac{4}{x}.$$

a)
Sketch the graph $y = f(x)$.

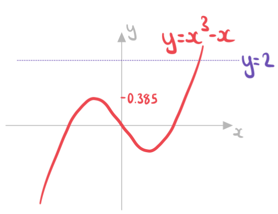
Use GDC to find max, min, intercepts




b)
Write down the number of real solutions to the equation $x^3 - x = 2$.

Identify the number of intersections between

$y = x^3 - x$ and $y = 2$




1 intersection

1 solution

c)
Find the coordinates of the points where $y = f(x)$ and $y = g(x)$ intersect.

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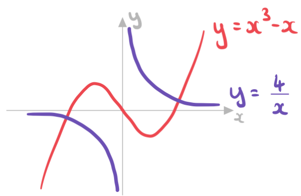
Use GDC to sketch both graphs




(−1.60, −2.50) and (1.60, 2.50)

d)

Write down the solutions to the equation $x^3 - x = \frac{4}{x}$.

Solutions to $x^3 - x = \frac{4}{x}$ are the $x$ coordinates of the points of intersection.

$x = -1.60$ and $x = 1.60$

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## 2.4 Further Functions & Graphs

### 2.4.1 Reciprocal & Rational Functions

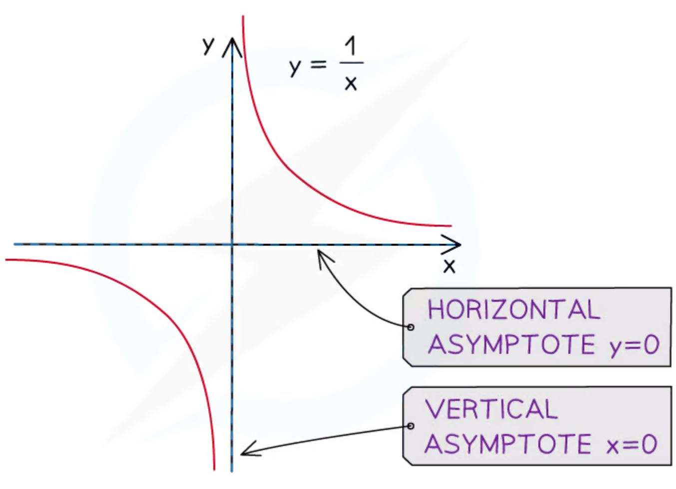
# Reciprocal Functions & Graphs

## What is the reciprocal function?

- The **reciprocal function** is defined by $f(x) = \frac{1}{x},\ x \neq 0$
- Its **domain** is the set of **all real values except 0**
- Its **range** is the set of **all real values except 0**
- The reciprocal function has a **self-inverse** nature
  - $f^{-1}(x) = f(x)$
  - $(f \circ f)(x) = x$

## What are the key features of the reciprocal graph?

- The graph **does not have a *y*-intercept**
- The graph **does not have any roots**
- The graph has **two asymptotes**
  - A **horizontal** asymptote at the *x*-axis: $y = 0$
    - This is the **limiting value** when the absolute value of *x* gets very large
  - A **vertical** asymptote at the *y*-axis: $x = 0$
    - This is the value that causes the **denominator to be zero**
- The graph has **two axes of symmetry**
  - $y = x$
  - $y = -x$
- The graph **does not have any minimum or maximum points**



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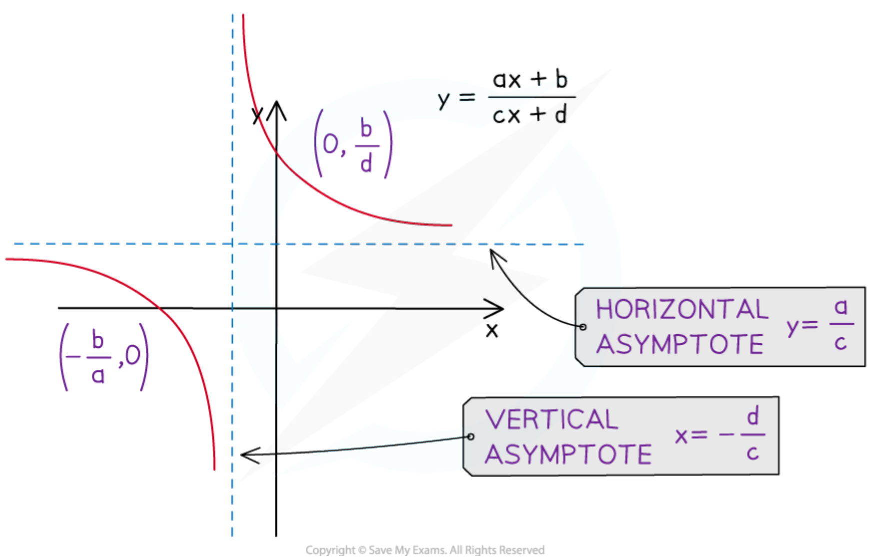
## Linear Rational Functions & Graphs

### What is a rational function?

- A **rational function** is of the form $f(x) = \frac{ax+b}{cx+d}, x \neq -\frac{d}{c}$
- Its **domain** is the set of **all real values except** $-\frac{d}{c}$
- Its **range** is the set of **all real values except** $\frac{a}{c}$
- The **reciprocal function** is a **special case** of a rational function

### What are the key features of rational graphs?

- The graph has a **y-intercept** at $\left(0, \frac{b}{d}\right)$ provided $d \neq 0$
- The graph has **one root** at $\left(-\frac{b}{a}, 0\right)$ provided $a \neq 0$
- The graph has **two asymptotes**
    - A **horizontal** asymptote: $y = \frac{a}{c}$
        - This is the **limiting value** when the absolute value of *x* gets very large
    - A **vertical** asymptote: $x = -\frac{d}{c}$
        - This is the value that causes the **denominator to be zero**
- The graph **does not have any minimum or maximum points**
- If you are asked to **sketch or draw** a rational graph:
    - Give the **coordinates** of any **intercepts** with the axes
    - Give the **equations** of the **asymptotes**



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### Exam Tip

- If you draw a horizontal line anywhere it should only intersect this type of graph once at most
- The only horizontal line that should not intersect the graph is the horizontal asymptote
  - This can be used to check your sketch in an exam

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## Worked Example

The function $f$ is defined by $f(x) = \dfrac{10 - 5x}{x + 2}$ for $x \neq -2$.

a)
Write down the equation of
(i)
the vertical asymptote of the graph of $f$,
(ii)
the horizontal asymptote of the graph of $f$.

(i) Vertical asymptote is when denominator equals zero

$x + 2 = 0$ $\quad \boxed{x = -2}$

(ii) Horizontal asymptote is limiting value as $x$ gets large

$\lim_{x\to\infty} \dfrac{10-5x}{x+2} = \lim_{x\to\infty} \dfrac{-5x}{x}$ $\quad \boxed{y = -5}$

b)
Find the coordinates of the intercepts of the graph of $f$ with the axes.

y-intercept occurs when $x = 0$

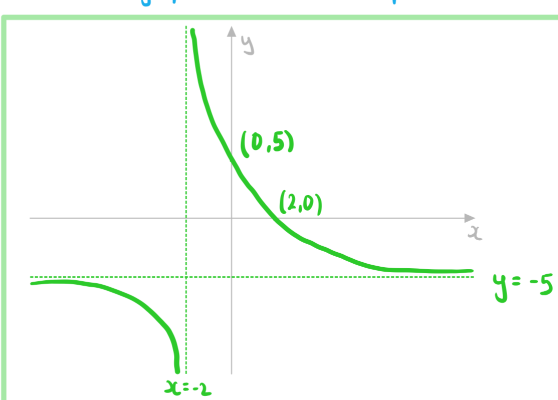
$y = \dfrac{10 - 5(0)}{0 + 2} = 5$ $\quad \boxed{(0, 5)}$

x-intercept occurs when $y = 0$

$\dfrac{10 - 5x}{x + 2} = 0 \Rightarrow 10 - 5x = 0 \Rightarrow x = 2$ $\quad \boxed{(2, 0)}$

c)
Sketch the graph of $f$.

Include asymptotes and intercepts



2.4.2 Exponential & Logarithmic Functions

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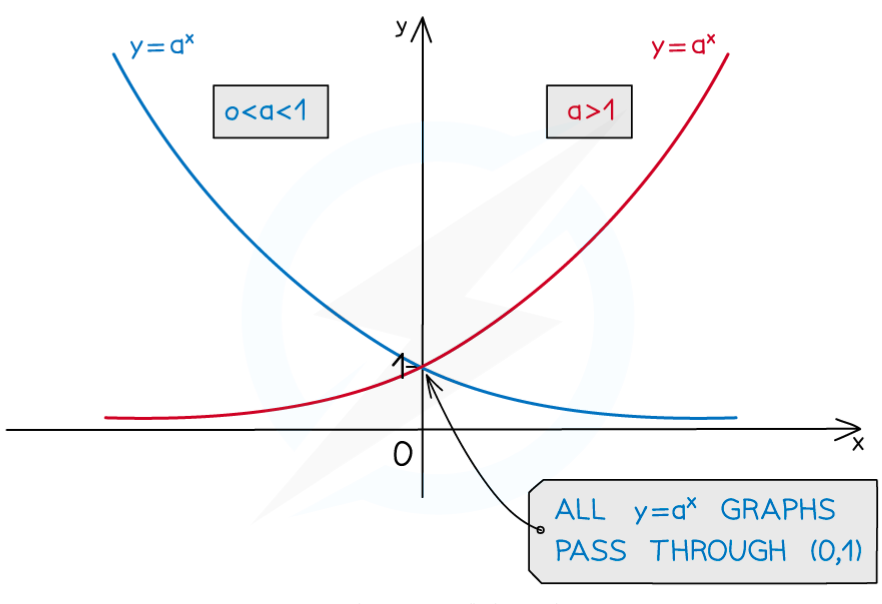
## Exponential Functions & Graphs

### What is an exponential function?

- An **exponential function** is defined by $f(x) = a^x,\ a > 0$
- Its **domain** is the set of **all real values**
- Its **range** is the set of **all positive real values**
- An important exponential function is $f(x) = \mathrm{e}^x$
  - Where e is the mathematical constant 2.718...
- Any exponential function can be written using e
  - $a^x = \mathrm{e}^{x\ln a}$
    - This is given in the **formula booklet**

### What are the key features of exponential graphs?

- The graphs have a **y-intercept** at (0, 1)
- The graphs **do not have any roots**
- The graphs have **a horizontal asymptote** at the *x*-axis: $y = 0$
  - For ***a* > 1** this is the **limiting value** when *x* tends to **negative infinity**
  - For **0 < *a* < 1** this is the **limiting value** when *x* tends to **positive infinity**
- The graphs **do not have any minimum or maximum points**



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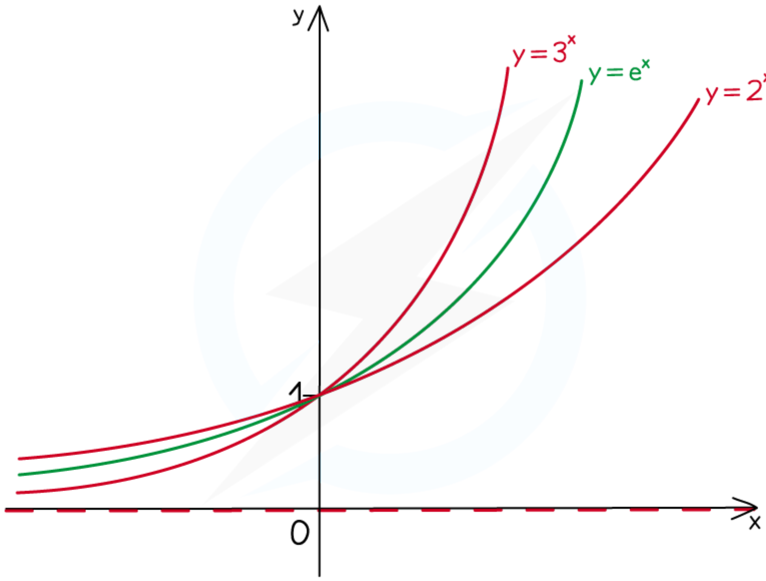




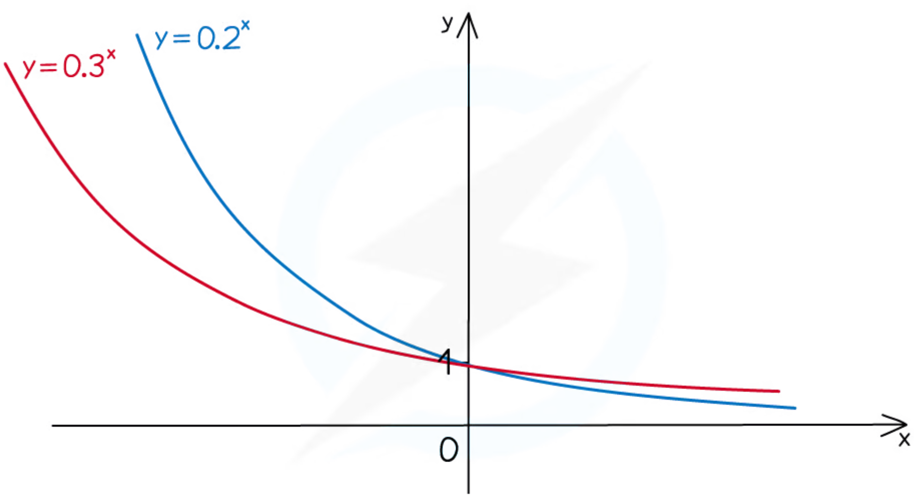

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# Logarithmic Functions & Graphs

## What is a logarithmic function?

- A **logarithmic function** is of the form $f(x) = \log_a x,\ x > 0$
- Its **domain** is the set of all **positive real values**
  - You can't take a log of zero or a negative number
- Its **range** is set of **all real values**
- $\log_a x$ and $a^x$ are **inverse** functions
- An important logarithmic function is $f(x) = \ln x$
  - This is the natural logarithmic function $\ln x \equiv \log_{\mathrm{e}} x$
  - This is the inverse of $\mathrm{e}^x$
    - $\ln \mathrm{e}^x = x$ and $\mathrm{e}^{\ln x} = x$
- Any logarithmic function can be written using ln
  - $\log_a x = \dfrac{\ln x}{\ln a}$ using the change of base formula

## What are the key features of logarithmic graphs?

- The graphs **do not have a *y*-intercept**
- The graphs have **one root** at (1, 0)
- The graphs have a **vertical asymptote** at the *y*-axis: $x = 0$
- The graphs **do not have any minimum or maximum points**

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## Worked Example

The function $f$ is defined by $f(x) = \log_5 x$ for $x > 0$.

a)
Write down the inverse of $f$. Give your answer in the form $e^{g(x)}$.

Formula booklet

| | |
|---|---|
| Exponents & logarithms | $a^x = b \Leftrightarrow x = \log_a b \qquad a > 0, b > 0, a \neq 1$ |

$x = \log_5 y \quad \Leftrightarrow \quad y = 5^x$

Formula booklet

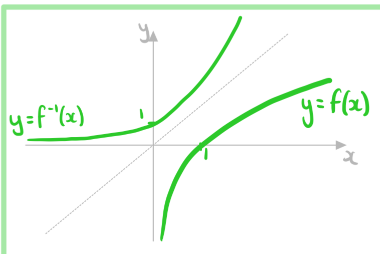
| | |
|---|---|
| Exponential & logarithmic functions | $a^x = e^{x \ln a}$ |

$5^x = e^{x\ln 5}$

$f^{-1}(x) = e^{x\ln 5}$

b)
Sketch the graphs of $f$ and its inverse on the same set of axes.

$f$ and $f^{-1}$ are reflections in line $y = x$



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## 2.4.3 Solving Equations

### Solving Equations Analytically

### How can I solve equations analytically where the unknown appears only once?

- These equations can be **solved by rearranging**
- For **one-to-one functions** you can just apply the **inverse**
  - Addition and subtraction are inverses
    - $y = x + k \Leftrightarrow x = y - k$
  - Multiplication and division are inverses
    - $y = kx \Leftrightarrow x = \frac{y}{k}$
  - Taking the reciprocal is a self-inverse
    - $y = \frac{1}{x} \Leftrightarrow x = \frac{1}{y}$
  - Odd powers and roots are inverses
    - $y = x^n \Leftrightarrow x = \sqrt[n]{y}$
    - $y = x^n \Leftrightarrow x = y^{\frac{1}{n}}$
  - Exponentials and logarithms are inverses
    - $y = a^x \Leftrightarrow x = \log_a y$
    - $y = e^x \Leftrightarrow x = \ln y$
- For **many-to-one functions** you will need to use your knowledge of the functions to find the **other solutions**
  - Even powers lead to positive and negative solutions
    - $y = x^n \Leftrightarrow x = \pm\sqrt[n]{y}$
  - Modulus functions lead to positive and negative solutions
    - $y = |x| \Leftrightarrow x = \pm y$
  - Trigonometric functions lead to infinite solutions using their symmetries
    - $y = \sin x \Leftrightarrow x = 2k\pi + \sin^{-1} y \text{ or } x = (1 + 2k)\pi - \sin^{-1} y$
    - $y = \cos x \Leftrightarrow x = 2k\pi \pm \cos^{-1} y$
    - $y = \tan x \Leftrightarrow x = k\pi + \tan^{-1} y$
- Take care when you apply **many-to-one functions** to **both sides** of an equation as this can create **additional solutions** which are incorrect
  - For example: squaring both sides
    - $x + 1 = 3$ has one solution $x = 2$
    - $(x + 1)^2 = 3^2$ has two solutions $x = 2$ and $x = -4$
- Always **check your solutions** by substituting back into the **original equation**

### How can I solve equations analytically where the unknown appears more than once?

- Sometimes it is possible to **simplify expressions** to make the **unknown appear only once**
- **Collect all terms** involving $x$ on **one side** and try to simplify into one term
  - For **exponents** use
    - $a^{f(x)} \times a^{g(x)} = a^{f(x) + g(x)}$

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- $\frac{a^{f(x)}}{a^{g(x)}} = a^{f(x)-g(x)}$
- $(a^{f(x)})^{g(x)} = a^{f(x)\times g(x)}$
- $a^{f(x)} = e^{f(x)\ln a}$
  - For **logarithms** use
    - $\log_a f(x) + \log_a g(x) = \log_a(f(x)\times g(x))$
    - $\log_a f(x) - \log_a g(x) = \log_a\left(\frac{f(x)}{g(x)}\right)$
    - $n\log_a f(x) = \log_a (f(x))^n$

## How can I solve equations analytically when the equation can't be simplified?

- Sometimes it is **not possible to simplify** equations
- Most of these equations **cannot be solved analytically**
- A **special case** that can be solved is where the equation can be **transformed into a quadratic** using a substitution
  - These will have **three terms** and involve the same type of function
- **Identify** the **suitable substitution** by considering which **function is a square of another**
  - For example: the following can be transformed into $2y^2 + 3y - 4 = 0$
    - $2x^4 + 3x^2 - 4 = 0$ using $y = x^2$
    - $2x + 3\sqrt{x} - 4 = 0$ using $y = \sqrt{x}$
    - $\frac{2}{x^6} + \frac{3}{x^3} - 4 = 0$ using $y = \frac{1}{x^3}$
    - $2e^{2x} + 3e^x - 4 = 0$ using $y = e^x$
    - $2\times 25^x + 3\times 5^x - 4 = 0$ using $y = 5^x$
    - $2^{2x+1} + 3\times 2^x - 4 = 0$ using $y = 2^x$
    - $2(x^3-1)^2 + 3(x^3-1) - 4 = 0$ using $y = x^3 - 1$
- To **solve**:
  - Make the **substitution** $y = f(x)$
  - **Solve** the **quadratic equation** $ay^2 + by + c = 0$ to get $y_1$ & $y_2$
  - **Solve** $f(x) = y_1$ and $f(x) = y_2$
    - Note that some equations might have **zero or several solutions**

## Can I divide both sides of an equation by an expression?

- When dividing by an expression you must consider whether the **expression could be zero**
- Dividing by an expression that could be zero could result in you **losing solutions to the original equation**
  - For example: $(x+1)(2x-1) = 3(x+1)$
    - If you divide both sides by $(x+1)$ you get $2x - 1 = 3$ which gives $x = 2$
    - However $x = -1$ is also a solution to the original equation
- To ensure you **do not lose solutions** you can:
  - **Split the equation into two equations**
    - One where the dividing expression equals zero: $x + 1 = 0$
    - One where the equation has been divided by the expression: $2x - 1 = 3$
  - **Make the equation equal zero and factorise**

- $(x+1)(2x-1)-3(x+1)=0$
- $(x+1)(2x-1-3)=0$ which gives $(x+1)(2x-4)=0$
- Set each factor equal to zero and solve: $x+1=0$ and $2x-4=0$

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### Exam Tip

- A common mistake that students make in exams is applying functions to each term rather than to each side
  - For example: Starting with the equation $\ln x + \ln(x-1) = 5$ it would be incorrect to write $e^{\ln x} + e^{\ln(x-1)} = e^5$ or $x + (x-1) = e^5$
  - Instead it would be correct to write $e^{\ln x + \ln(x-1)} = e^5$ and then simplify from there

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## Worked Example

Find the exact solutions for the following equations:

a)
$5 - 2\log_4 x = 0$.

Rearrange using inverse functions

$5 - 2\log_4 x = 0$

$2\log_4 x = 5$ ($y = x - k \Leftrightarrow x = y + k$)

$\log_4 x = \frac{5}{2}$ ($y = kx \Leftrightarrow x = \frac{y}{k}$)

$x = 4^{\frac{5}{2}}$ ($y = \log_a x \Leftrightarrow x = a^y$)

$x = (\sqrt{4})^5$ ($a^{\frac{m}{n}} = (\sqrt[n]{a})^m$)

$x = 32$

b)
$x = \sqrt{x+2}$.

Square both sides (Many-to-one function)

$x^2 = x + 2 \Rightarrow x^2 - x - 2 = 0$

$(x-2)(x+1) = 0 \Rightarrow x = 2$ or $x = -1$

Check whether each solution is valid

$x = 2$: LHS $= 2$ RHS $= \sqrt{2+2} = 2$ ✓

$x = -1$: LHS $= -1$ RHS $= \sqrt{-1+2} = 1$ ✗

$x = 2$

c)
$e^{2x} - 4e^x - 5 = 0$.

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Notice $e^{2x} = (e^x)^2$, let $y = e^x$

$y^2 - 4y - 5 = 0 \Rightarrow (y+1)(y-5) = 0$

$y = -1$ or $y = 5$

Solve using $y = e^x$

$e^x = -1$ has no solutions as $e^x > 0$

$e^x = 5 \quad \therefore \quad x = \ln 5$

$\boxed{x = \ln 5}$

## Solving Equations Graphically

YOUR NOTES

### How can I solve equations graphically?

- To solve $f(x) = g(x)$
  - One method is to **draw the graphs** $y = f(x)$ and $y = g(x)$
    - The **solutions** are the **x-coordinates** of the points of **intersection**
  - Another method is to **draw the graph** $y = f(x) - g(x)$ or $y = g(x) - f(x)$
    - The **solutions** are the **roots (zeros)** of this graph
      - This method is sometimes quicker as it involves **drawing only one graph**

### Why do I need to solve equations graphically?

- Some equations **cannot be solved analytically**
  - **Polynomials** of degree higher than 4
    - $x^5 - x + 1 = 0$
  - Equations involving **different types of functions**
    - $e^x = x^2$

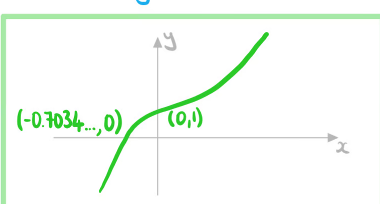
**Exam Tip**

- On a calculator paper you are allowed to solve equations using your GDC unless the question asks for an algebraic method
- If your answer needs to be an exact value then you might need to solve analytically to get the exact value

**Worked Example**

a)
Sketch the graph $y = e^x - x^2$.

Sketch using GDC




b)
Hence find the solution to $e^x = x^2$.

$e^x = x^2$ when $e^x - x^2 = 0$

Solution is the $x$-intercept of $y = e^x - x^2$

$x = -0.703$ (3sf)

## 2.4.4 Modelling with Functions

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### Modelling with Functions

### What is a mathematical model?

- A **mathematical model** simplifies a real-world situation so it can be described using mathematics
  - The model can then be used to make predictions
- **Assumptions** about the situation are made in order to simplify the mathematics
- Models can be **refined** (improved) if further information is available or if the model is compared to real-world data

### How do I set up the model?

- The question could:
  - give you the equation of the model
  - tell you about the relationship
    - It might say the relationship is linear, quadratic, etc
  - ask you to suggest a **suitable model**
    - Use your knowledge of each model
    - E.g. if it is compound interest then an exponential model is the most appropriate
- You may have to determine a **reasonable domain**
  - Consider real-life context
    - E.g. if dealing with hours in a day then
    - E.g. if dealing with physical quantities (such as length) then
  - Consider the **possible ranges**
    - If the outcome cannot be negative then you want to choose a domain which corresponds to a range with no negative values
    - **Sketching the graph** is helpful to determine a suitable domain

### Which models might I need to use?

- You could be given any model and be expected to use it
- Common models include:
  - **Linear**
    - Arithmetic sequences
    - Linear regression
  - **Quadratic**
    - Projectile motion
    - The height of a cable supporting a bridge
    - Profit
  - **Exponential**
    - Geometric sequences
    - Exponential growth and decay
    - Compound interest
  - **Logarithmic**
    - Richter scale for the magnitude of earthquakes
  - **Rational**
    - Temperature of a cup of coffee

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- **Trigonometric**
  - The depth of a tide

## How do I use a model?

- You can use a model by substituting in values for the variable to **estimate outputs**
  - For example: Let $h(t)$ be the height of a football $t$ seconds after being kicked
    - $h(3)$ will be an estimate for the height of the ball 3 seconds after being kicked
- Given an **output** you can **form an equation** with the model to **estimate the input**
  - For example: Let $P(n)$ be the profit made by selling $n$ items
    - Solving $P(n) = 100$ will give you an estimate for the number of items needing to be sold to make a profit of 100
- If your variable is **time** then substituting $t = 0$ will give you the **initial value** according to the model
- Fully understand the **units for the variables**
  - If the units of $P$ are measured in **thousand dollars** then $P = 3$ represents $3000
- Look out for **key words** such as:
  - Initially
  - Minimum/maximum
  - Limiting value

## What do I do if some of the parameters are unknown?

- A general method is to **form equations** by substituting in given values
  - You can form **multiple equations** and **solve them simultaneously** using your GDC
  - This method **works for all models**
- The **initial value** is the value of the function when the variable is 0
  - This is **normally one of the parameters** in the equation of the model

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## Worked Example

The temperature, $T$°C, of a cup of coffee is monitored. Initially the temperature is 80°C and 5 minutes later it is 40°C . It is suggested that the temperature follows the model:

$$T(t) = Ae^{kt} + 16,\ t \geq 0.$$

where $t$ is the time, in minutes, after the coffee has been made.

a)
State the value of $A$.

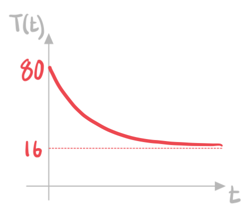
Initially temperature is 80°C

$T(0) = 80$

$Ae^{-k(0)} + 16 = 80$

$A + 16 = 80$

$A = 64$




b)
Find the exact value of $k$.

$t = 5$ , $T = 40$

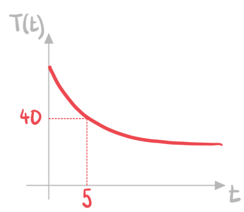
$40 = 64e^{5k} + 16$

$64e^{5k} = 24$

$e^{5k} = \frac{3}{8}$

$5k = \ln\frac{3}{8}$

$k = \frac{1}{5}\ln\frac{3}{8}$




c)
Find the time taken for the temperature of the coffee to reach 30°C.

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Find $t$ such that $T(t) = 30$

$30 = 64e^{kt} + 16$

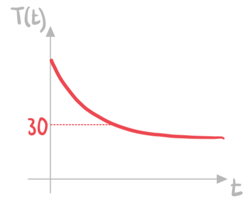
Leave as $k$ until the end to save writing $\frac{1}{5}\ln\frac{3}{8}$ each time

$64e^{kt} = 14$

$e^{kt} = \frac{7}{32}$

$kt = \ln\frac{7}{32}$

$t = \frac{\ln\frac{7}{32}}{k} = \frac{\ln\frac{7}{32}}{\frac{1}{5}\ln\frac{3}{8}} = 7.7476...$




**7.75 minutes (3sf)**

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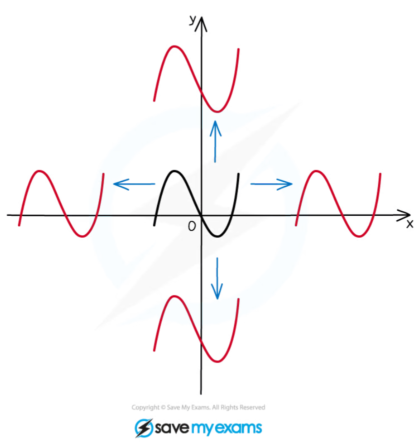
# 2.5 Transformations of Graphs

## 2.5.1 Translations of Graphs

### Translations of Graphs

### What are translations of graphs?

- When you alter a function in certain ways, the effects on the graph of the function can be described by **geometrical transformations**
- For a **translation**:
  - the graph is **moved** (up or down, left or right) in the xy plane
    - Its position **changes**
  - the shape, size, and orientation of the graph remain **unchanged**
- A particular translation (how far left/right, how far up/down) is specified by a **translation vector** $\begin{pmatrix} x \\ y \end{pmatrix}$:
  - *x* is the **horizontal** displacement
    - **Positive** moves **right**
    - **Negative** moves **left**
  - *y* is the **vertical** displacement
    - **Positive** moves **up**
    - **Negative** moves **down**




### What effects do horizontal translations have on the graphs and functions?

- A **horizontal translation** of the graph $y = f(x)$ by the vector $\begin{pmatrix} a \\ 0 \end{pmatrix}$ is represented by
  - $y = f(x - a)$
- The ***x*-coordinates change**
  - The value *a* is **subtracted** from them
- The ***y*-coordinates stay the same**
- The coordinates $(x, y)$ become $(x - a, y)$
- **Horizontal** asymptotes **stay the same**

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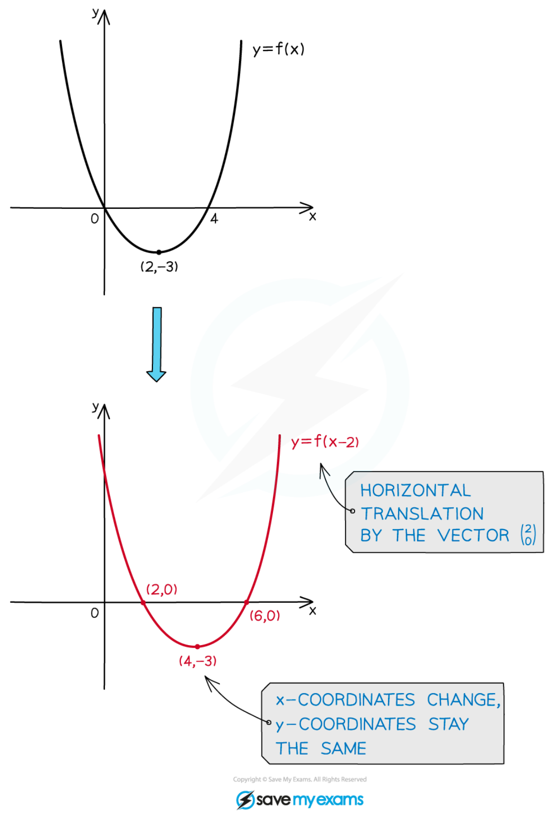
- **Vertical** asymptotes **change**
  - $x = k$ becomes $x = k - a$




## What effects do vertical translations have on the graphs and functions?

- A **vertical translation** of the graph $y = f(x)$ by the vector $\begin{pmatrix} 0 \\ b \end{pmatrix}$ is represented by
  - $y - b = f(x)$
  - This is often rearranged to $y = f(x) + b$
- The ***x*-coordinates stay the same**
- The ***y*-coordinates change**
  - The value *b* is **added** to them
- The coordinates $(x, y)$ become $(x, y + b)$
- **Horizontal** asymptotes **change**
  - $y = k$ becomes $y = k + b$
- **Vertical** asymptotes **stay the same**

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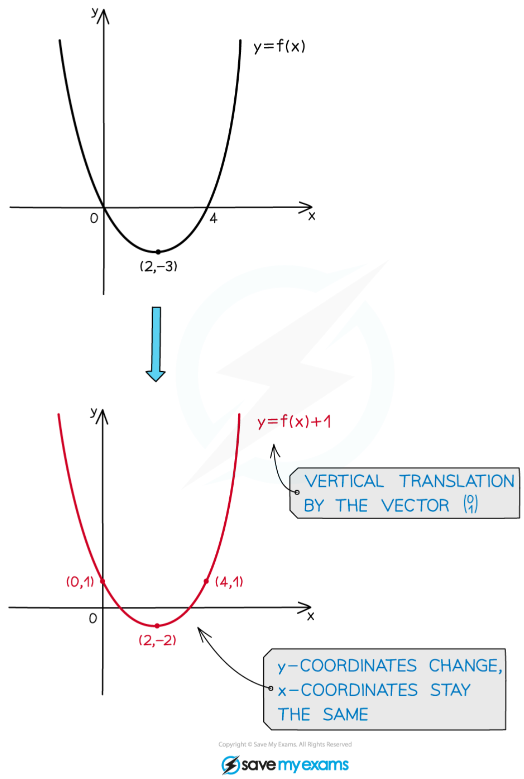


## Exam Tip

- To get full marks in an exam make sure you use correct mathematical terminology
  - For example: Translate by the vector $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$

YOUR NOTES
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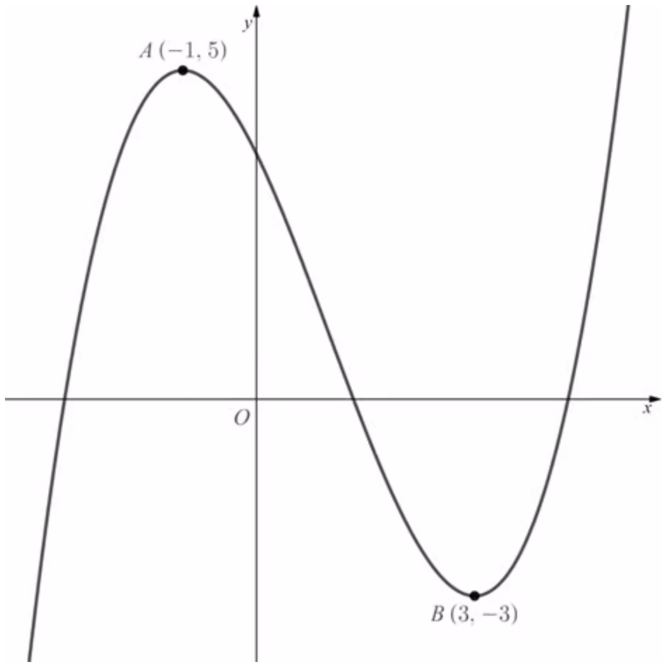
## Worked Example

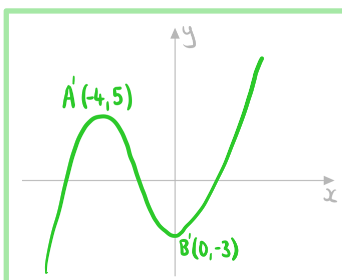
The diagram below shows the graph of $y = f(x)$.




a)
Sketch the graph of $y = f(x+3)$.

$y = f(x+k)$ translation by $\begin{pmatrix} -k \\ 0 \end{pmatrix}$

Translate $y=f(x)$ by $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$
A becomes $(-4, 5)$
B becomes $(0, -3)$




b)
Sketch the graph of $y = f(x) + 3$.

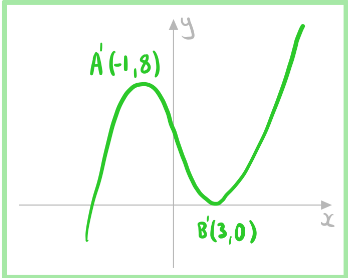
YOUR NOTES

$y = f(x) + k$ translation by $\begin{pmatrix} 0 \\ k \end{pmatrix}$

Translate $y = f(x)$ by $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$

A becomes $(-1, 8)$

B becomes $(3, 0)$



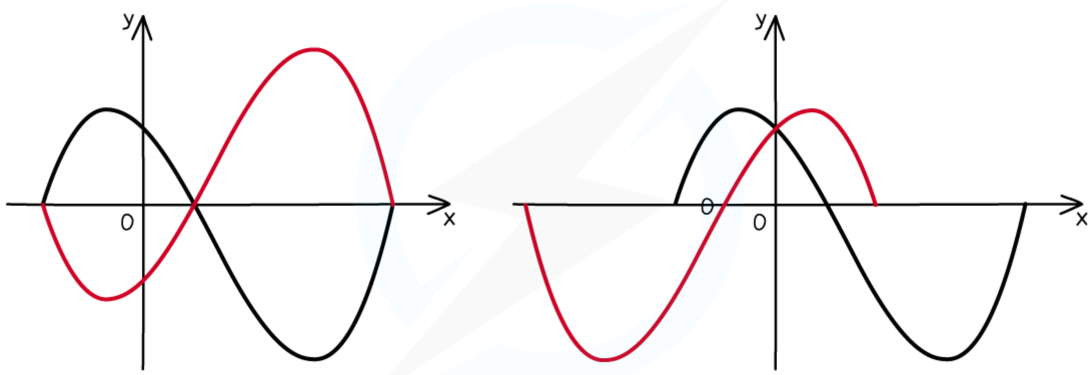
2.5.2 Reflections of Graphs

YOUR NOTES

## Reflections of Graphs

### What are reflections of graphs?

- When you alter a function in certain ways, the effects on the graph of the function can be described by **geometrical transformations**
- For a **reflection**:
  - the graph is **flipped** about one of the coordinate axes
    - Its orientation **changes**
  - the size of the graph remains **unchanged**
- A particular reflection is specified by an **axis of symmetry**:
  - $y = 0$
    - This is the *x*-axis
  - $x = 0$
    - This is the *y*-axis





### What effects do horizontal reflections have on the graphs and functions?

- A **horizontal reflection** of the graph $y = f(x)$ about the *y*-axis is represented by
  - $y = f(-x)$
- The ***x*-coordinates change**
  - Their **sign** changes
- The ***y*-coordinates stay the same**
- The coordinates $(x, y)$ become $(-x, y)$
- **Horizontal** asymptotes **stay the same**
- **Vertical** asymptotes **change**
  - $x = k$ becomes $x = -k$

YOUR NOTES
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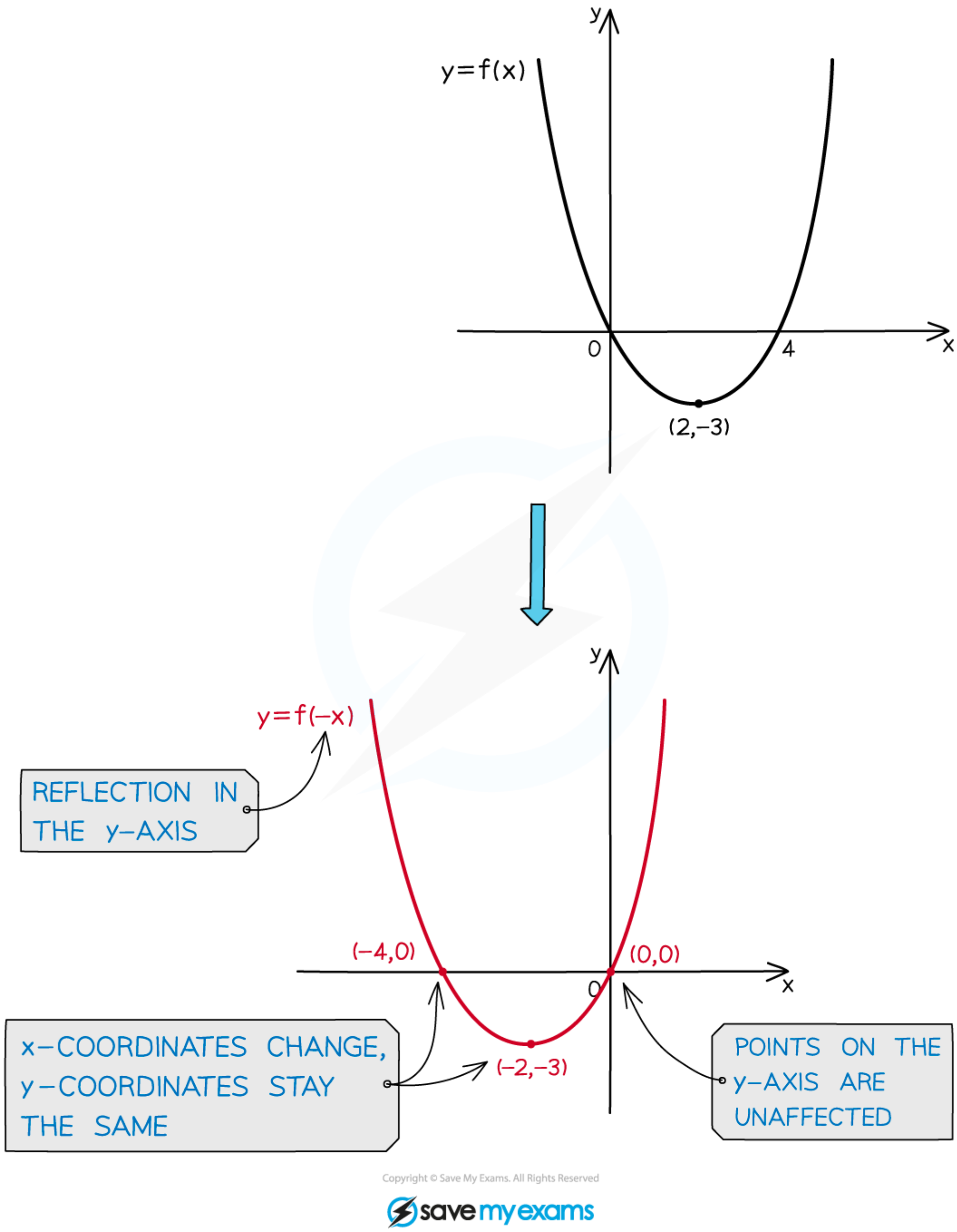


## What effects do vertical reflections have on the graphs and functions?

- A **vertical reflection** of the graph $y = f(x)$ about the *x*-axis is represented by
  - $-y = f(x)$
  - This is often rearranged to $y = -f(x)$
- The **x-coordinates stay the same**
- The ***y*-coordinates change**
  - Their **sign** changes

- The coordinates $(x, y)$ become $(x, -y)$
- **Horizontal** asymptotes **change**
  - $y = k$ becomes $y = -k$
- **Vertical** asymptotes **stay the same**

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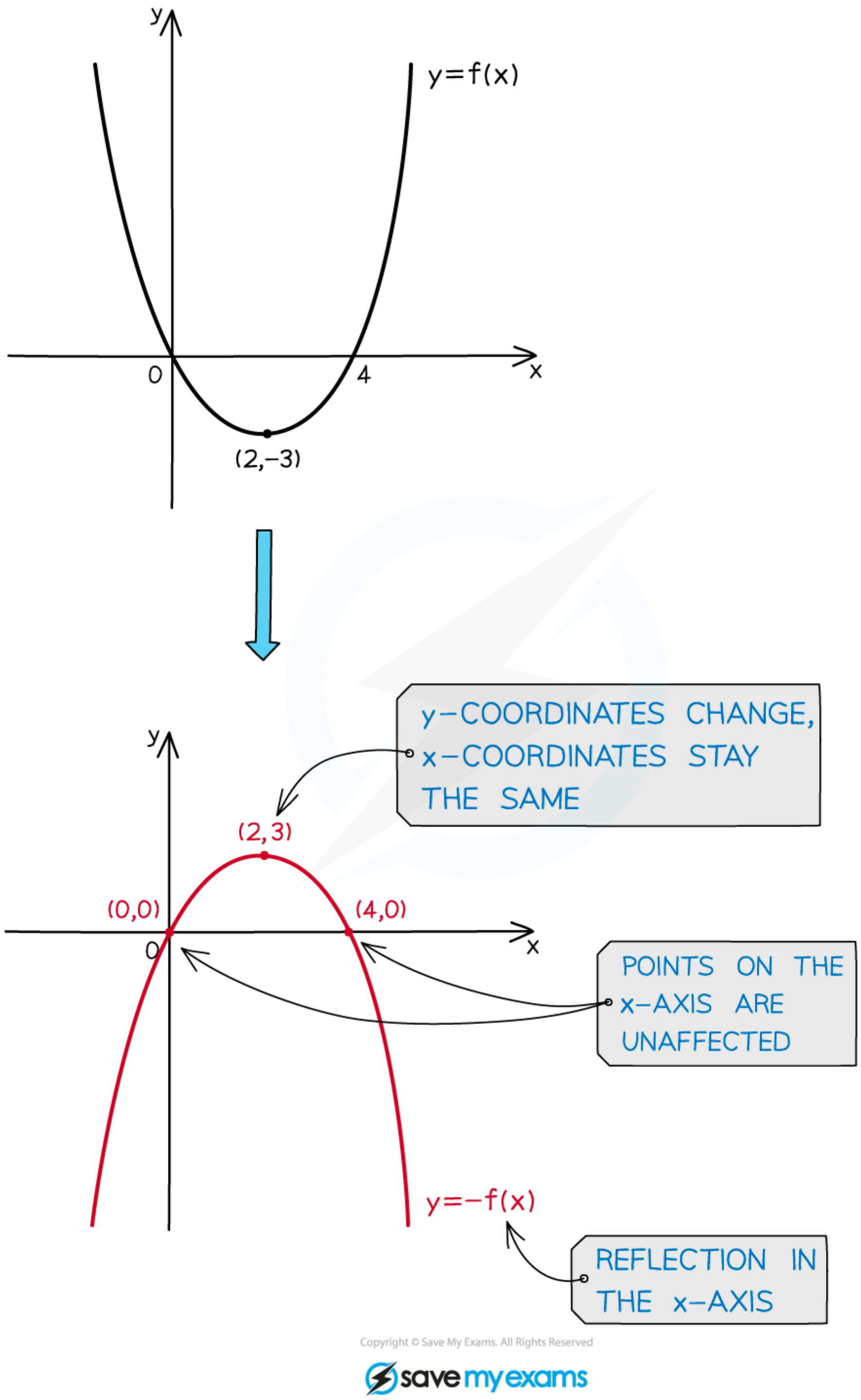

YOUR NOTES
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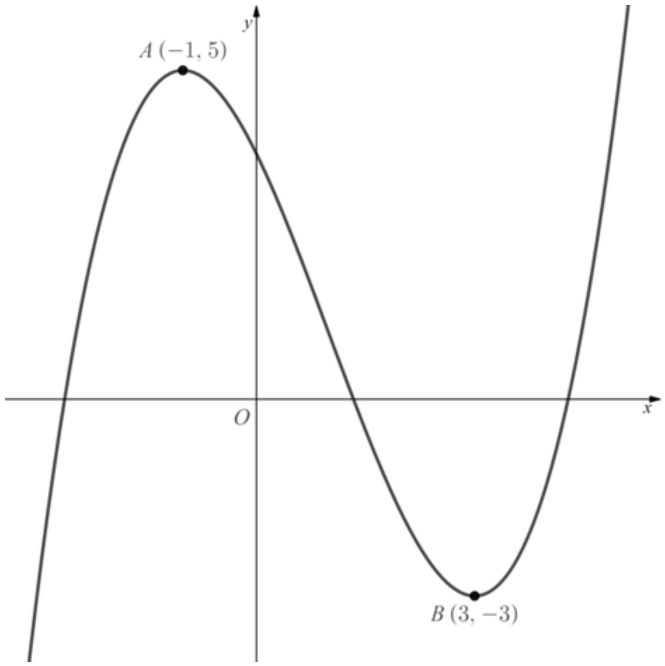
## Worked Example

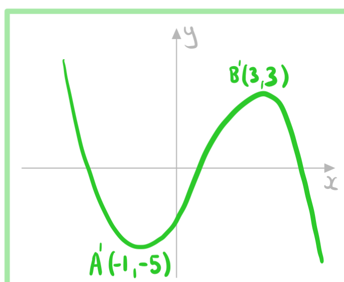
The diagram below shows the graph of $y = f(x)$.




a)
Sketch the graph of $y = -f(x)$.

y = - f(x) reflection in x-axis

A becomes (-1, -5)
B becomes (3, 3)

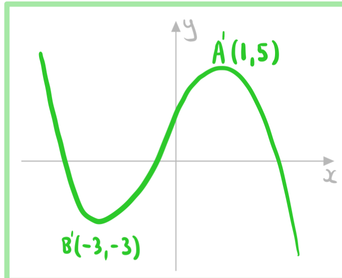



b)
Sketch the graph of $y = f(-x)$.

$y = f(-x)$ reflection in $y$-axis

A becomes $(1, 5)$

B becomes $(-3, -3)$




YOUR NOTES

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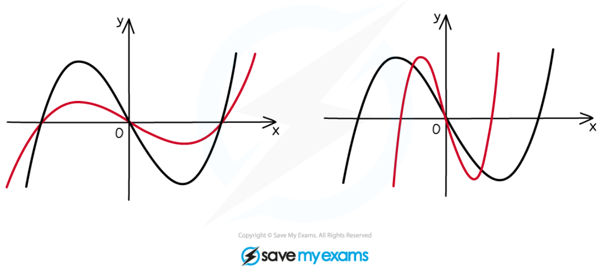
## 2.5.3 Stretches of Graphs

# Stretches of Graphs

## What are stretches of graphs?

- When you alter a function in certain ways, the effects on the graph of the function can be described by **geometrical transformations**
- For a **stretch**:
  - the graph is **stretched** about one of the coordinate axes by a scale factor
    - Its size **changes**
  - the orientation of the graph remains **unchanged**
- A particular stretch is specified by a **coordinate axis** and a **scale factor**:
  - The **distance** between a **point** on the graph and the **specified coordinate axis** is **multiplied** by the **constant scale factor**
  - The graph is stretched in the **direction** which is **parallel** to the **other coordinate axis**
  - For scale factors **bigger than 1**
    - the points on the graph get **further away** from the **specified coordinate axis**
  - For scale factors **between 0 and 1**
    - the points on the graph get **closer** to the **specified coordinate axis**
    - This is also called a **compression**




## What effects do horizontal stretches have on the graphs and functions?

- A **horizontal stretch** of the graph $y = f(x)$ by a scale factor *q* centred about the *y*-axis is represented by
  - $y = f\left(\frac{x}{q}\right)$
- The ***x*-coordinates change**
  - They are **divided** by *q*
- The ***y*-coordinates stay the same**
- The coordinates $(x, y)$ become $\left(\frac{x}{q}, y\right)$
- **Horizontal** asymptotes **stay the same**
- **Vertical** asymptotes **change**
  - $x = k$ becomes $x = \frac{k}{q}$

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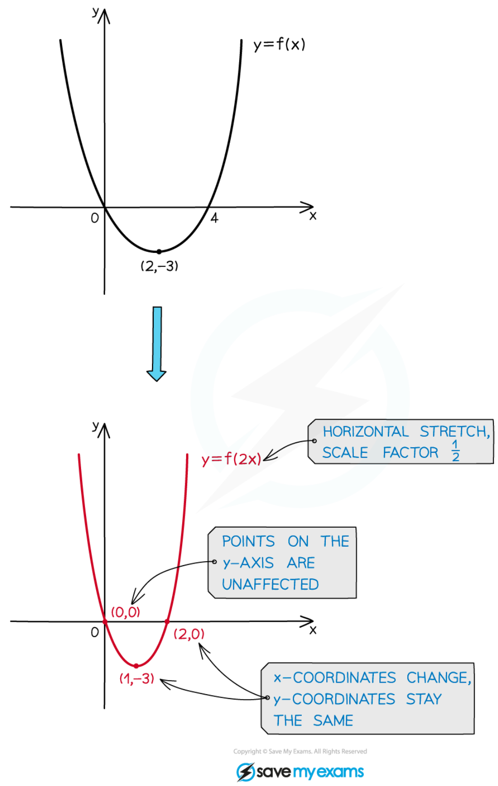
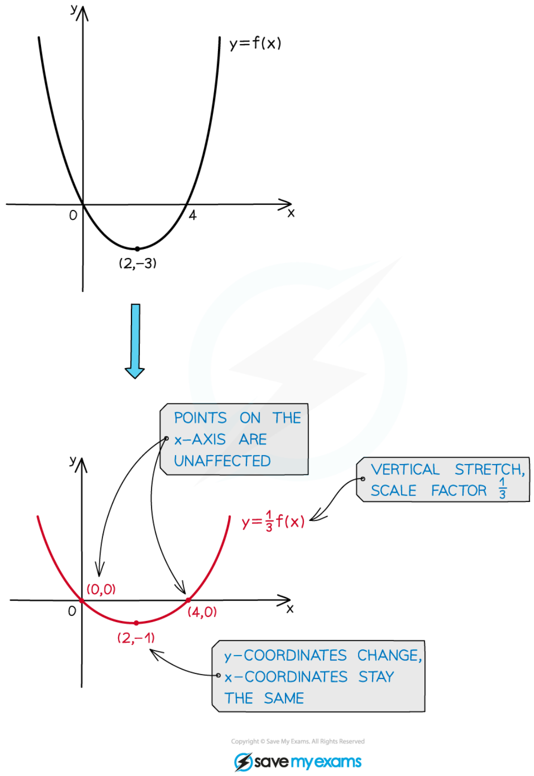



## What effects do vertical stretches have on the graphs and functions?

- A **vertical stretch** of the graph $y = f(x)$ by a scale factor *p* centred about the *x*-axis is represented by
  - $\frac{y}{p} = f(x)$
  - This is often rearranged to $y = pf(x)$
- The ***x*-coordinates stay the same**
- The ***y*-coordinates change**
  - They are **multiplied** by *p*
- The coordinates $(x, y)$ become $(x, py)$
- **Horizontal** asymptotes **change**
  - $y = k$ becomes $y = pk$
- **Vertical** asymptotes **stay the same**




YOUR NOTES

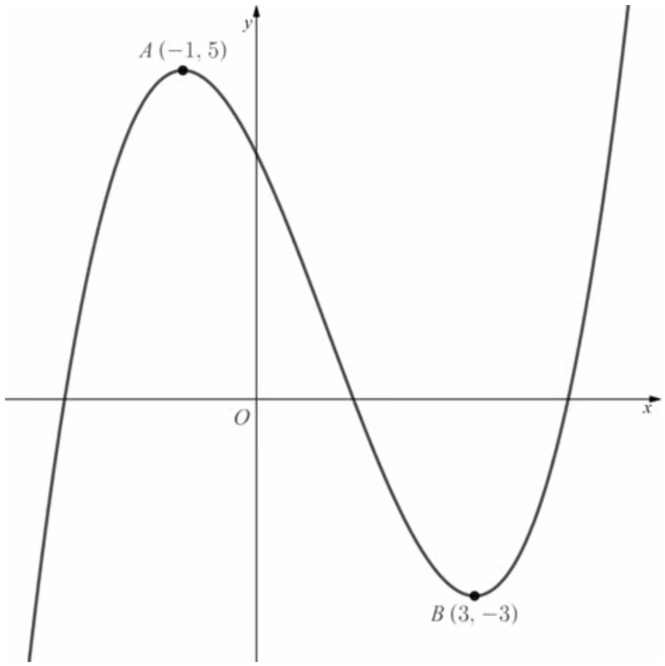
## Exam Tip

- To get full marks in an exam make sure you use correct mathematical terminology
  - For example: Stretch vertically by scale factor ½
  - Do not use the word "compress" in your exam

YOUR NOTES

## Worked Example

The diagram below shows the graph of $y = f(x)$.

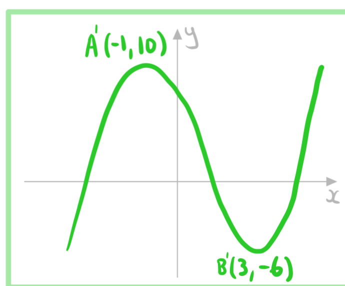



a)
Sketch the graph of $y = 2f(x)$.

$y = kf(x)$ vertical stretch scale factor $k$

Stretch $y = f(x)$ vertically scale factor 2

A becomes $(-1, 10)$
B becomes $(3, -6)$

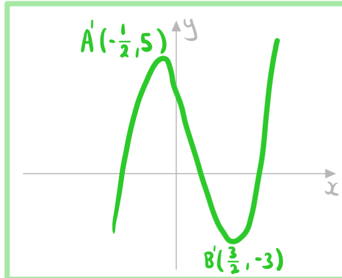



b)
Sketch the graph of $y = f(2x)$.

$y = f(kx)$ horizontal stretch scale factor $\frac{1}{k}$

Stretch $y = f(x)$ horizontally scale factor $\frac{1}{2}$

A becomes $(-\frac{1}{2}, 5)$
B becomes $(\frac{3}{2}, -3)$




YOUR NOTES

2.5.4 Composite Transformations of Graphs

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## Composite Transformations of Graphs

### What transformations do I need to know?

- $y = f(x+k)$ is **horizontal translation** by vector $\begin{pmatrix} -k \\ 0 \end{pmatrix}$
  - If *k* is **positive** then the graph moves **left**
  - If *k* is **negative** then the graph moves **right**
- $y = f(x)+k$ is **vertical translation** by vector $\begin{pmatrix} 0 \\ k \end{pmatrix}$
  - If *k* is **positive** then the graph moves **up**
  - If *k* is **negative** then the graph moves **down**
- $y = f(kx)$ is a **horizontal stretch** by scale factor $\frac{1}{k}$ centred about the *y*-axis
  - If $k > 1$ then the graph gets **closer** to the *y*-axis
  - If $0 < k < 1$ then the graph gets **further** from the *y*-axis
- $y = kf(x)$ is a **vertical stretch** by scale factor $k$ centred about the *x*-axis
  - If $k > 1$ then the graph gets **further** from the *x*-axis
  - If $0 < k < 1$ then the graph gets **closer** to the *x*-axis
- $y = f(-x)$ is a **horizontal reflection** about the *y*-axis
  - A **horizontal reflection** can be viewed as a special case of a **horizontal stretch**
- $y = -f(x)$ is a **vertical reflection** about the *x*-axis
  - A **vertical reflection** can be viewed as a special case of a **vertical stretch**

### How do horizontal and vertical transformations affect each other?

- **Horizontal and vertical transformations** are **independent** of each other
  - The horizontal transformations involved will need to be applied in their correct order
  - The vertical transformations involved will need to be applied in their correct order
- Suppose there are **two horizontal** transformation **$H_1$ then $H_2$** and **two vertical** transformations **$V_1$ then $V_2$** then they can be applied in the following orders:
  - Horizontal then vertical:
    - $H_1H_2V_1V_2$
  - Vertical then horizontal:
    - $V_1V_2H_1H_2$
  - Mixed up (provided that **$H_1$ comes before $H_2$** and **$V_1$ comes before V2**):
    - $H_1V_1H_2V_2$
    - $H_1V_1V_2H_2$
    - $V_1H_1V_2H_2$
    - $V_1H_1H_2V_2$

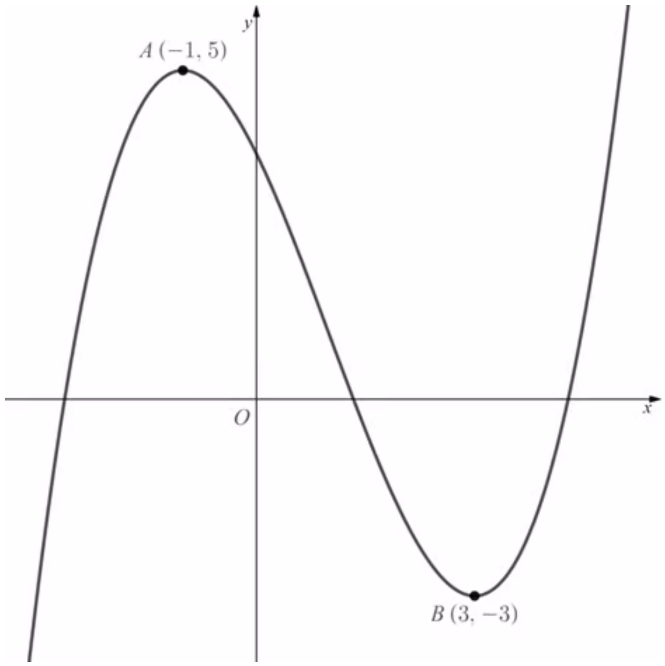
**Exam Tip**

- In an exam you are more likely to get the correct solution if you deal with one transformation at a time and sketch the graph after each transformation

YOUR NOTES

## Worked Example

The diagram below shows the graph of $y = f(x)$.




Sketch the graph of $y = \frac{1}{2} f\left(\frac{x}{2}\right)$.

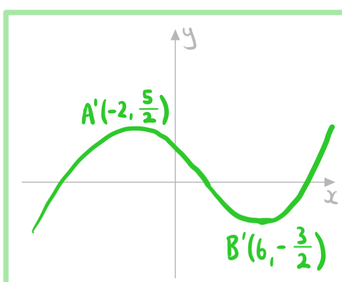
A vertical and horizontal transformation can be done in any order

$y = \frac{1}{2}f(x)$ : vertical stretch scale factor $\frac{1}{2}$

$y = f\left(\frac{x}{2}\right)$ : horizontal stretch scale factor 2

A becomes $\left(-2, \frac{5}{2}\right)$

B becomes $\left(6, -\frac{3}{2}\right)$



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## Composite Vertical Transformations af(x)+b

### How do I deal with multiple vertical transformations?

- **Order matters** when you have **more than one vertical transformations**
- If you are asked to find the equation then **build up the equation** by looking at the transformations in order
  - A **vertical stretch** by scale factor *a* followed by a **translation** of $\begin{pmatrix} 0 \\ b \end{pmatrix}$
    - Stretch: $y = af(x)$
    - Then translation: $y = [af(x)] + b$
    - Final equation: $y = af(x) + b$
  - A **translation** of $\begin{pmatrix} 0 \\ b \end{pmatrix}$ followed by a **vertical stretch** by scale factor *a*
    - Translation: $y = f(x) + b$
    - Then stretch: $y = a[f(x) + b]$
    - Final equation: $y = af(x) + ab$
- If you are asked to determine the **order**
  - The order of vertical transformations **follows the order of operations**
  - First write the equation in the form $y = af(x) + b$
    - **First stretch vertically** by scale factor *a*
    - If *a* is negative then the **reflection and stretch** can be **done in any order**
    - **Then translate** by $\begin{pmatrix} 0 \\ b \end{pmatrix}$

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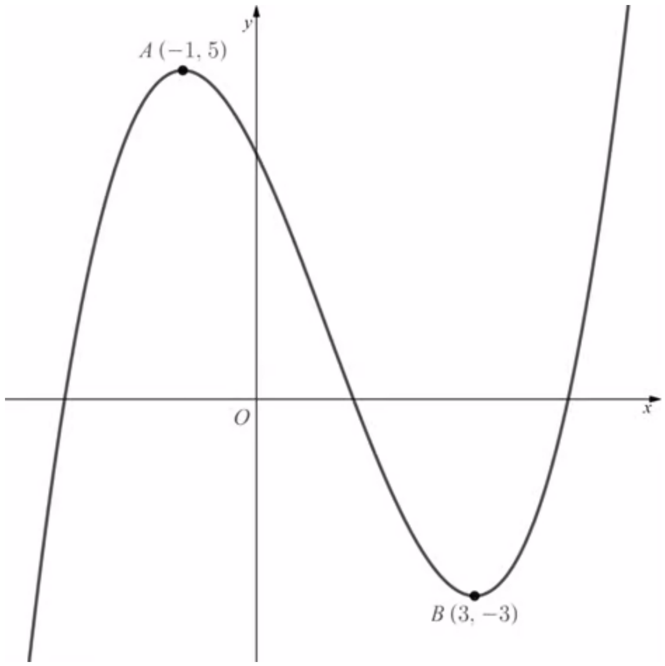
## Worked Example

The diagram below shows the graph of $y = f(x)$.




Sketch the graph of $y = 3f(x) - 2$.

The order vertical transformations follows the order of operations

$y = 3f(x)$ : Vertical stretch scale factor 3

$y = f(x) - 2$ : Translate $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$

A becomes (−1, 13)

B becomes (3, −11)