6.2 Extended Questions (Section B, HL)

Question Paper

Course	DP IB Maths
Section	6. Extended Questions
Topic	6.2 Extended Questions (Section B, HL)
Difficulty	Hard

Time allowed: 120

Score: /93

Percentage: /100

Question 1a

The points A(2, 3, 0), B(-2, 4, 1), C(1, -1, 3) and D(5, -2, 2) lie on the plane Π_1 and form a parallelogram, where AB and CD are one pair of parallel edges and BC and AD are the other pair of parallel edges. Each unit on the coordinate grid is equivalent to 1 cm in length.

a)
Find the vector product of $\overrightarrow{\mathrm{AB}}$ and $\overrightarrow{\mathrm{AC}}$.

[3 marks]

Question 1b

b)
Hence, or otherwise, find the Cartesian equation of the plane Π_1.

[2 marks]

Question 1c

A second plane Π_2 contains the point with position vector $\begin{pmatrix} 5 \\ -3 \\ 5 \end{pmatrix}$ and also the line L, which has vector equation

$$\boldsymbol{r} = \begin{pmatrix} 6 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix}.$$

c)
Show that Π_1 and Π_2 are parallel.

[4 marks]

Question 1d

A parallelepiped is a 3D object made up of six faces that are parallelograms lying in pairs of parallel planes. EFGH is a parallelogram on Π_2 that is congruent to ABCD, and points A, B, C and D on Π_1 are joined to points E, F, G and H respectively on Π_2 to form a parallelepiped.

d)
Given that the coordinates of E are (3, 6, 0), find the coordinates of point H.

[3 marks]

Question 1e

The volume of a parallelepiped can be found using the formula $|(\boldsymbol{a} \times \boldsymbol{b}).\boldsymbol{c}|$ where $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are vectors corresponding to three edges meeting at a single vertex of the parallelepiped.

e)
Show that the volume of the parallelepiped ABCDEFGH is 40 cm^3.

[5 marks]

Question 2a

A function g is defined by $g(x) = \arccos\left(\frac{x^2 - 1}{x^2 + 1}\right),\ x \in \mathbb{R}$.

a)
Show that g is an even function.

[1 mark]

Question 2b

b)
By considering the limit of g as x tends to infinity, show that the graph of $y = g(x)$ has a horizontal asymptote and state its equation.

[2 marks]

Question 2c

c)
(i)
Show that $g'(x) = \frac{-2x}{\left(\sqrt{x^2}\right)(x^2+1)}$ for $x \in \mathbb{R},\ x \geq 0$.

(ii)
Considering the fact that $\sqrt{x^2} = |x|$, and also the expression for $g'(x)$ above, show that g is increasing for $x < 0$.

[9 marks]

Question 2d

A new function, h, is created by restricting the domain of g, such that $h(x) = \arccos\left(\frac{x^2 - 1}{x^2 + 1}\right),\ x \in \mathbb{R},\ x \geq 0.$, , .

d)
Find an expression for $h^{-1}(x)$, carefully considering the range of h in determining your final answer.

[5 marks]

Question 2e

e)
State the domain of $h^{-1}(x)$.

[2 marks]

Question 3a

The function f is defined by $f(x) = \dfrac{4x+3}{9x^2-4}$, for $x \in \mathbb{R},\ x \neq p,\ x \neq q.$

a)
Given that $p < q$, find the value of p and the value of q.

[2 marks]

Question 3b

b)
Find an expression for $f'(x)$.

[3 marks]

Question 3c

The graph of $y = f(x)$ has exactly one point of inflection.

c)
Find the x-coordinate of the point of inflection.

[2 marks]

Question 3d

d)
Sketch the graph of $y = f(x)$ for $-3 \leq x \leq 3$, showing the values of any axes intercepts, the coordinates of any local maxima and local minima, and giving the equations of any asymptotes.

[5 marks]

Question 3e

The function g is defined by $g(x) = \dfrac{9x^2 - 4}{4x + 3}$, for $x \in \mathbb{R}$, $x \neq -\dfrac{3}{4}$.

e)
Find the equations of all the asymptotes on the graph of $y = g(x)$.

[4 marks]

Question 3f

f)
By considering the graph of $y = f(x) - g(x)$, or otherwise, solve $f(x) < g(x)$ for $x \in \mathbb{R}$.

[4 marks]

Question 4a

The derivative of the function f is given by $f'(x) = \dfrac{1}{x(k-x)}$, $x \in \mathbb{R},\ x \neq 0,\ x \neq k$, where $k > 0$ is a real constant.

a)
By finding appropriate constants a and b in terms of k, show that the expression for $f'(x)$ can be written in the form $\dfrac{a}{x} + \dfrac{b}{k-x}$, where $a, b \in \mathbb{R}$.

[3 marks]

Question 4b

b)
Hence find an expression for $f(x)$.

[3 marks]

Question 4c

Consider a population of lizards, P, which has an initial size of 800. The rate of change of the population can be modelled by the differential equation $\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{P(k-P)}{25k}$, where t is the time measured in years, $t \geq 0$, and k is the maximum sustainable population.

c)
By solving the differential equation, show that

$$P = \frac{800k}{(k-800)e^{-\frac{t}{25}} + 800}$$

[8 marks]

Question 4d

At $t = 12$ the lizard population has reduced in size to three fourths of its original value.

d)
Find the value of k, giving your answer correct to four significant figures.

[3 marks]

Question 4e

e)
Find the value of t when the population is decreasing at a rate of 16 lizards per year.

[3 marks]

Question 5a

A mathematical function f is defined by $f(x) = xe^{2x}$.

a)
Show that $f''(x) = (4x + 4)e^{2x}$.

[3 marks]

Question 5b

b)
Prove by mathematical induction that if $f(x) = xe^{2x}$ then $f^{(n)}(x) = (2^n x + n2^{n-1})e^{2x}$.

[7 marks]

Question 5c

Let $g(x) = \ln(1 + mx),\ m \in Z^+$.

Consider the function h defined by $h(x) = f(x) \times g(x)$.

c)
Given that the term in x^4 of the Maclaurin series for $h(x)$ has coefficient 6, find the value of m.

[7 marks]