

# 6.2 Extended Questions (Paper 2, HL)

## Question Paper

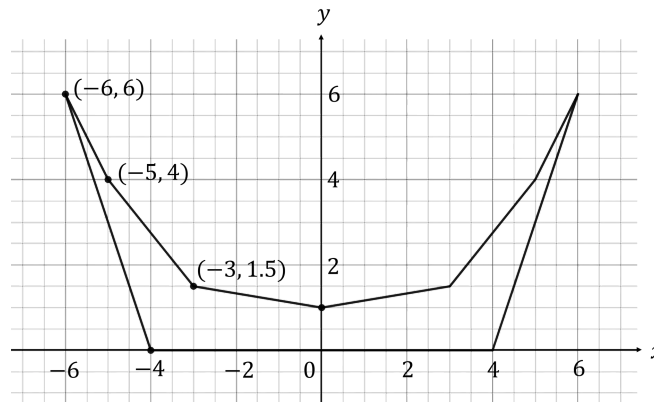
Course	DPIB Maths
Section	6. Extended Questions
Topic	6.2 Extended Questions (Paper 2, HL)
Difficulty	Medium

**Time allowed:** 100  
**Score:** /83  
**Percentage:** /100

### Question 1a

Paul finds an unusually shaped bowl when excavating his garden. It appears to be made out of bronze, and Paul decides to model the shape in order to work out its volume.

By uploading a photograph of the object onto some graphing software, Paul identifies that the cross-section of the bowl goes through the points  $(-4,0)$ ,  $(-6,6)$ ,  $(-5,4)$ ,  $(-3,1.5)$  and  $(0,1)$ . The cross-section is symmetrical about the  $y$ -axis as shown in the diagram. All of the units are in centimetres.



He models the section from  $(-4,0)$  to  $(-6,6)$  as a straight line.

- a)  
Find the equation of the line passing through these two points.

[2 marks]

### Question 1b

Paul models the section of the bowl that passes through the points  $(-6,6)$ ,  $(-5,4)$ ,  $(-3,1.5)$  and  $(0,1)$  with a quadratic curve.

- b)  
(i)  
Find the equation of the least squares quadratic curve for these four points.

- (ii)  
By considering the gradient of this curve when  $x = -0.5$ , explain why it may not be a good model.

[1 mark]

**Question 1c**

Paul thinks that a quadratic with a minimum at  $(0, 1)$  and passing through the point  $(-6, 6)$  is a better option.

c)

Find the equation of the new model.

[1 mark]

**Question 1d**

Believing this to be a better model for the bowl, Paul finds the volume of revolution about the  $y$ -axis to estimate the volume of the bowl.

d)

Re-arrange the answers to parts (a) and (c) to make  $x$  a function of  $y$ .

[1 mark]

**Question 1e**

e)

(i)

Write down an expression for Paul's estimate of the volume as the difference of two integrals.

(ii)

Hence find the value of Paul's estimate.

[1 mark]

### Question 2a

A boat is moving such that its position vector when viewed from above at time  $t$  seconds can be modelled by

$$r = \begin{pmatrix} 10 - a \sin\left(\frac{\pi t}{600}\right) \\ b\left(1 - \cos\left(\frac{\pi t}{600}\right)\right) \end{pmatrix}$$

with respect to a rectangular coordinate system from a point  $O$ , where the non-zero constants  $a$  and  $b$  can be determined. All distances are given in metres.

The boat leaves its mooring point at time  $t = 0$  seconds and 5 minutes later is at the point with coordinates  $(-20, 40)$ .

a)

Find

(i)

the values of  $a$  and  $b$ ,

(ii)

the displacement of the boat from its mooring point.

[4 marks]

### Question 2b

b)

Find the velocity vector of the boat at time  $t$  seconds.

[2 marks]

**Question 2c**

After setting off, the boat reaches a point P where it is moving parallel to the  $x$ -axis.

c)

Find OP.

[6 marks]

**Question 2d**

d)

Find the time that the boat returns to its mooring point and the acceleration of the boat at this moment.

[3 marks]

**Question 3a**

On a particular island, a particular species of bird was initially recorded as having a population of 80 at the start of a programme of observations. Over time, the scientists conducting the programme determined that the growth rate of the bird population could be modelled by the following differential equation

$$\frac{dx}{dt} = \frac{7}{5}x$$

where  $x$  is the size of the bird population, and  $t$  is the length of time in years since the start of the programme.

a)

Find the population of the bird species two years after the start of the programme.

**[5 marks]**

### Question 3b

When the population of the bird species reaches 2000, a new reptile species is introduced to the island in order to control the bird population. Initially 280 reptiles are introduced to the island. Based on their research the scientists believe that the interaction between the two species after the introduction of the reptiles can be modelled by the system of coupled differential equations

$$\frac{dx}{dt} = (3 - 0.012y)x$$

$$\frac{dy}{dt} = (0.0007x - 1)y$$

Where  $x$  and  $y$  represent the size of the bird and reptile populations respectively.

(b) Using the Euler method with a step size of 0.5, find an estimate for

- (i) the bird population 2 years after the reptiles were introduced
- (ii) the reptile population 2 years after the reptiles were introduced.

[6 marks]

### Question 3c

c)

Explain how the approximation in part (b) could be improved.

[1 mark]

### Question 3d

d)  
Show that the origin is an equilibrium point for the system, and determine the coordinates of the other equilibrium point.

[3 marks]

### Question 4a

In a game, enemies appear independently and randomly at an average rate of 2.5 enemies every minute.

a)  
Find the probability that exactly 3 enemies will appear during one particular minute.

[1 mark]

### Question 4b

b)  
Find the probability that exactly 10 enemies will appear in a five-minute period.

[2 marks]

### Question 4c

c)  
Find the probability that at least 3 enemies will appear in a 90-second period.

[2 marks]



**Question 4d**

d)

The probability that at least one enemy appears in  $k$  seconds is 0.999. Find the value of  $k$  correct to 3 significant figures.

**[2 marks]****Question 4e**

e)

A 10-minute interval is divided into ten 1-minute periods (first minute, second minute, third minute, etc.). Find the probability that there will be exactly two of those 1-minute periods in which no enemies appear.

**[4 marks]**

### Question 4f

On the next level of the game, there is a boss enemy and a number of additional henchmen to fight against.

The number of times that the boss enemy appears in a one-minute period can be modelled by a Poisson distribution with a mean of 1.1.

The number of times that an individual henchman appears in a one-minute period can be modelled by a Poisson distribution with a mean of 0.6.

It may be assumed that the boss enemy and the henchmen each appear randomly and independently of one another.

Each time that the boss enemy or any particular henchman appears, it is counted as one 'enemy appearance'.

f)

Determine the least number of henchmen required in order that the probability of 40 or more 'enemy appearances' occurring in a 3-minute period is greater than 0.38. You may assume that neither the boss enemy nor any of the henchmen are able to be totally eliminated from the game during this 3-minute period.

[4 marks]

### Question 5a

James throws a ball to his friend Mia. The height,  $h$ , in metres, of the ball above the ground is modelled by the function

$$h(t) = -1.05t^2 + 3.84t + 1.97, \quad t \geq 0$$

where  $t$  is the time, in seconds, from the moment that James releases the ball.

a)

Write down the height of the ball when James releases it.

[1 mark]

### Question 5b

After 4 seconds the ball is at a height of metres above the ground.

b)

Find the value of  $q$ .

[2 marks]

### Question 5c

c)

Find  $h'(t)$

[2 marks]

### Question 5d

d)

Find the maximum height reached by the ball and write down the corresponding time  $t$ .

[3 marks]

**Question 5e**

James then drives a remote-controlled car in a straight horizontal line from a starting position right in front of his feet. The velocity of the remote-controlled car in  $\text{ms}^{-1}$  is given by the equation

$$v(t) = \frac{5}{4}t^2 - \frac{19}{2}t + 18t - 2$$

e)

Find an expression for the horizontal displacement of the remote-controlled car from its starting position at time  $t$  seconds.

**[4 marks]****Question 5f**

f)

Find the total horizontal distance that the remote-controlled car has travelled in the first 5 seconds.

**[3 marks]**

**Question 6a**

Consider the following system of differential equations:

$$\frac{dx}{dt} = x + 2y$$

$$\frac{dy}{dt} = -3x - 4y$$

(a)

Find the eigenvalues and corresponding eigenvectors of the matrix  $\begin{pmatrix} 1 & 2 \\ -3 & -4 \end{pmatrix}$ .

[6 marks]

**Question 6b**

(b)

Hence write down the general solution of the system.

[2 marks]

**Question 6c**

When  $t = 0$ ,  $x = 2$  and  $y = 4$ .

(c)

Use the given initial condition to determine the exact solution of the system.

[3 marks]

**Question 6d**

(d)

(i)

Find the value of  $\frac{dy}{dx}$  when  $t = 0$ .

(ii) Find the values of  $x, y$  and  $\frac{dy}{dx}$  when  $t = \ln \frac{9}{7}$ .

[3 marks]

**Question 6e**

e)

Hence sketch the solution trajectory of the system for  $t \geq 0$ .

[3 marks]

