# 5.7 Further Differential Equations

## Question Paper

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| Course | DP IB Maths |
| Section | 5. Calculus |
| Topic | 5.7 Further Differential Equations |
| Difficulty | Hard |

**Time allowed:** 120

**Score:** /96

**Percentage:** /100

## Question 1a

Consider the following system of coupled differential equations

$$x = -3x + e^{-4t}y$$

$$y = 6e^{-2t}x + y$$

with the initial condition $x = 1,\ y = 2$ when $t = 0$.

a)
Use the Euler method with a step size of 0.1 to find approximations for the values of $x$ and $y$ when $t = 0.5$.

**[6 marks]**

## Question 1b

b)
Show that the system has no equilibrium points other than the origin, for any value of $t$.

**[4 marks]**

## Question 2a

Consider the following system of differential equations:

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{2}x - 2y$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x - \frac{5}{2}y$$

a)

By first finding the eigenvalues and corresponding eigenvectors of an appropriate matrix, determine the general solution of the system.

**[7 marks]**

## Question 2b

When $t = 0,\ x = -3$ and $y = 2$.

b)
Use the given initial condition to determine the exact solution of the system.

**[3 marks]**

## Question 2c

c)
Describe the long-term behaviour of the variables $x$ and $y$.

**[2 marks]**

## Question 3a

The rates of change of two variables, $x$ and $y$, are described by the following system of differential equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 3x - 2y$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 3x - 4y$$

The matrix $\begin{pmatrix} 3 & -2 \\ 3 & -4 \end{pmatrix}$ has eigenvectors $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$. Initially $x = 7$ and $y = 1$.

a)
Use the above information to find the exact solution to the system of differential equations.

**[7 marks]**

## Question 3b

b)
Use the Euler method with a step size of 0.2 to find approximations for the values of $x$ and $y$ when $t = 1$.

**[6 marks]**

## Question 3c

c)

i)

Find the percentage error of the approximations from part (b) compared with the exact values of $x$ and $y$ when $t = 1$.

ii)

Explain how the approximations found in part (b) could be improved.

**[4 marks]**

## Question 4a

Consider a system of coupled differential equations with a general solution given by

$$x = Ae^{pt}\begin{pmatrix}2\\1\end{pmatrix} + Be^{qt}\begin{pmatrix}-2\\3\end{pmatrix}$$

where $p$ and $q$ are real constants.

For each of the relationships between $p$ and $q$ given below,

i)

sketch the phase portrait for the system

ii)

state whether the point  is a stable equilibrium point or an unstable equilibrium point.

(a) $p < q < 0.$

**[4 marks]**

## Question 4b

b) $p < 0 < q$

**[4 marks]**

## Question 4c

c) $0 < p < q$

**[4 marks]**

## Question 5a

The behaviour of two variables, $x$ and $y$, is modelled by the following system of differential equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 3x - 5y \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = x - y$$

where $x = 1$ and $y = 1$ when $t = 0$.

The matrix $\begin{pmatrix} 3 & -5 \\ 1 & -1 \end{pmatrix}$ has eigenvalues of $1 + \mathrm{i}$ and $1 - \mathrm{i}$.

(a) (i) Find the values of $\frac{\mathrm{d}x}{\mathrm{d}t}$ and $\frac{\mathrm{d}y}{\mathrm{d}t}$ at the point $(0,1)$.

(ii) Hence sketch the phase portrait of the system with the given initial condition.

**[4 marks]**

## Question 5b

It is suggested that the variables might better be described by the system

$$\frac{\mathrm{d}x}{\mathrm{d}y} = -3x - 5y \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = x + y$$

with the same initial conditions.

b) Calculate the eigenvalues of the matrix $\begin{pmatrix} -3 & -5 \\ 1 & 1 \end{pmatrix}$

**[3 marks]**

## Question 5c

c)

Hence describe how your phase portrait from part (a)(ii) would change to represent this new system of differential equations.

**[2 marks]**

## Question 6a

Scientists have been tracking levels, $x$ and $y$, of two atmospheric pollutants, and recording the levels of each relative to historical baseline figures (so a positive value indicates an amount higher than the baseline and a negative value indicates an amount less than the baseline). Based on known interactions of the pollutants with each other and with other substances in the atmosphere, the scientists propose modelling the situation with the following system of differential equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x - 2y$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = x - y$$

a) Find the values of $\frac{\mathrm{d}x}{\mathrm{d}t}$ and $\frac{\mathrm{d}y}{\mathrm{d}t}$ at the points $(1,0)$ and $(0,1)$.

**[2 marks]**

## Question 6b

b)

Find the eigenvalues of the matrix $\begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}$

**[3 marks]**

## Question 6c

At the start of the study both pollutants are above baseline levels, with $x = 5$ and $y = 3$.

c)
Use the above information to sketch a phase portrait showing the long-term behaviour of $x$ and $y$.

**[4 marks]**

## Question 7a

Scientists are studying populations of a prey species and a predator species within a particular region. They initially model the two species by the system of differential equations $\frac{\mathrm{d}x}{\mathrm{d}t} = 1.9x - 0.2y$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = 0.3x + 2.6y$, where $x$ represents the size of the prey population (in thousands) and $y$ represents the size of the predator population (in hundreds). Initially there are 2000 animals in the prey population and 450 in the predator population.

a)
Given that the eigenvalues of the matrix $\begin{pmatrix} 1.9 & -0.2 \\ 0.3 & 2.6 \end{pmatrix}$ are $2.5$ and 2, with corresponding eigenvectors $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$, sketch a possible trajectory for the change in the populations of the two animals over time.

**[4 marks]**

## Question 7b

Research suggests that neither species will disappear from the region in the foreseeable future.

b)
Criticise the model above, particularly in light of this research result.

**[1 mark]**

## Question 7c

It is suggested that the system of equations $\frac{dx}{dt} = (10 - 2y)x$ and $\frac{dy}{dt} = (3x - 6)y$ should be used as a model instead, where $t$ is measured in decades (1 decade= 10 years ).

c)
Determine the equilibrium points for the system under this model.

**[2 marks]**

## Question 7d

d)
i)
Use the Euler method with a step size of 0.002 to find approximations for the values of $x$ and $y$ at one-year intervals up to 8 years after the start of the study.
ii)
Use the values from (d)(i) to sketch a possible trajectory for the change in the populations of the two animals over time, and state what this suggests about the long-term behaviour of the two animal populations under the revised model.

**[9 marks]**

## Question 8a

A particle moves in a straight line, such that its displacement $x$ metres at time $t$ seconds is described by the differential equation

$$\frac{d^2x}{dt^2} + 7\frac{dx}{dt} + 13x = 109$$

where $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$ represent the particle's velocity and acceleration respectively.

a)

By letting $y = \frac{dx}{dt}$, show that the differential equation above can be written as a system of first order differential equations.

**[2 marks]**

## Question 8b

When $t = 0$, the displacement of the particle is zero and the velocity is $-2\ \text{ms}^{-1}$.

b)
By applying Euler's method with a step size of 0.1 to the system of equations found in part (a), along with the given initial condition, find approximations for the
i)
displacement
ii)
velocity
of the particle at time t = 0.5 .

**[6 marks]**

## Question 8c

c)
Use the Euler method to determine the long-term stable value of the particle's displacement.

**[2 marks]**

## Question 8d

d)

Use your answer from part (c) to explain why the long-term stable value of the particle's velocity must be zero.

**[1 mark]**