IB Maths DP

5. Calculus

YOUR NOTES
↓

CONTENTS

5.1 Differentiation

5.1.1 Introduction to Differentiation

Introduction to Derivatives

- Before introducing a **derivative**, an understanding of a **limit** is helpful

What is a limit?

- The **limit** of a **function** is the value a function (of x) approaches as x approaches a particular value from either side
  - Limits are of interest when the function is undefined at a particular value
  - For example, the function $\mathrm{f}(x) = \dfrac{x^4 - 1}{x - 1}$ will approach a limit as x approaches 1 from both below and above but is undefined at $x = 1$ as this would involve dividing by zero

What might I be asked about limits?

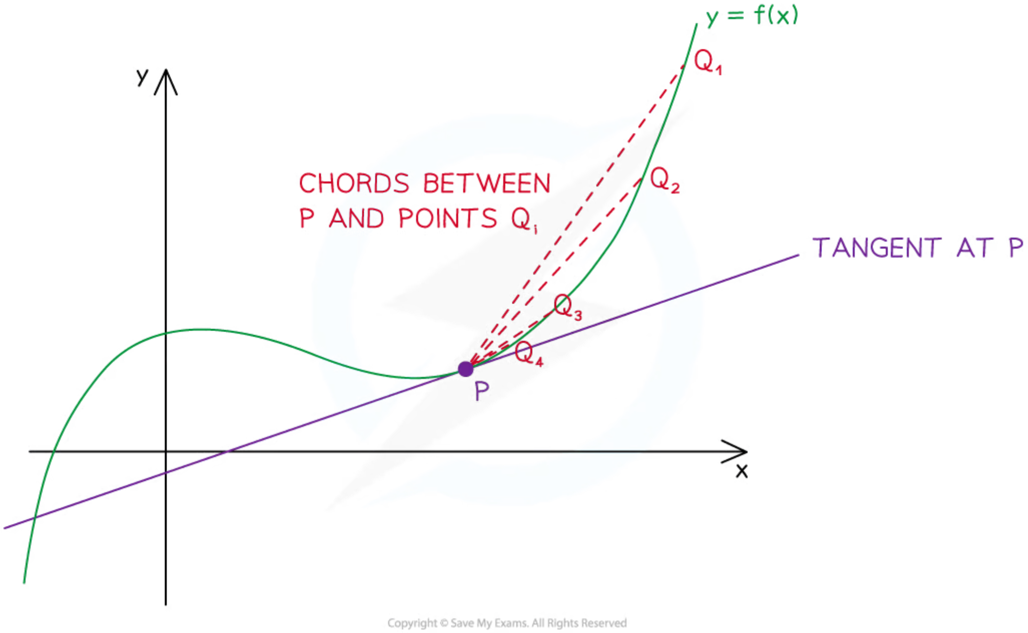
- You may be asked to predict or estimate limits from a table of function values or from the graph of $y = \mathrm{f}(x)$
- You may be asked to use your GDC to plot the graph and use values from it to estimate a limit

What is a derivative?

- **Calculus** is about **rates** of **change**
  - the way a car's position on a road changes is its speed
  - the way the car's speed changes is its acceleration
- The **gradient** (rate of change) of a (non-linear) **function** varies with x
- The **derivative** of a function is a function that relates the gradient to the value of x
- It is also called the **gradient function**

How are limits and derivatives linked?

- Consider the point P on the graph of $y = \mathrm{f}(x)$ as shown below
  - $[PQ_i]$ is a series of chords



- The **gradient** of the **function** $\mathrm{f}(x)$ at the point P is **equal** to the **gradient** of the **tangent** at point P
- The **gradient** of the **tangent** at point P is the **limit** of the **gradient** of the chords $[PQ_i]$ as point Q 'slides' down the curve and gets ever closer to point P
- The **gradient** of the function changes as x changes
- The **derivative** is the function that calculates the gradient from the value x

What is the notation for derivatives?

- For the function $y = \mathrm{f}(x)$ the **derivative**, with respect to x, would be written as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}'(x)$$

- Different variables may be used
  - e.g. If $V = \mathrm{f}(s)$ then $\frac{\mathrm{d}V}{\mathrm{d}s} = \mathrm{f}'(s)$

What might I be asked about derivatives?

- You may be asked to use the graphing features of your GDC to find the gradients of a function at different values of x
- From a series of gradient values, you may be asked to suggest an expression for the derivative (gradient function) of a function

Worked Example

The graph of $y = \mathrm{f}(x)$ where $\mathrm{f}(x) = x^3 - 2$ passes through the points $P(2,\ 6)$, $A(2.3,\ 10.167)$, $B(2.1,\ 7.261)$ and $C(2.05,\ 6.615125)$.

a)

Find the gradient of the chords $[PA]$, $[PB]$ and $[PC]$.

Gradient of a line (chord) is " $\frac{y_2 - y_1}{x_2 - x_1}$ "

$[PA]$: $\frac{10.167 - 6}{2.3 - 2} = 13.89$

$[PB]$: $\frac{7.261 - 6}{2.1 - 2} = 12.61$

$[PC]$: $\frac{6.615125 - 6}{2.05 - 2} = 12.3$

Gradient of chords are: $[PA]$ 13.89
$[PB]$ 12.61
$[PC]$ 12.3025

b)

Estimate the gradient of the tangent to the curve at the point P.

There will be a limit the gradient of the chord reaches as the difference in the x-coordinates approaches zero.

Estimate of gradient of tangent at $x = 2$ is 12

c)

Use your GDC to find the gradient of the tangent at the pont P.

Using GDC, plot $y = x^3 - 2$,
draw a tangent at $x = 2$
GDC can tell you either/both of the equation of the tangent and $\frac{dy}{dx}$

GDC gradient is 12

Differentiating Powers of x

What is differentiation?

- **Differentiation** is the process of finding an expression of the **derivative** (**gradient function**) from the expression of a function

How do I differentiate powers of x?

- **Powers** of x are **differentiated** according to the following formula:
  - If $\mathrm{f}(x) = x^n$ then $\mathrm{f}'(x) = nx^{n-1}$ where $n \in \mathbb{Z}$
  - This is given in the **formula booklet**
- If the power of x is **multiplied** by a **constant** then the derivative is also multiplied by that constant
  - If $\mathrm{f}(x) = ax^n$ then $\mathrm{f}'(x) = anx^{n-1}$ where $n \in \mathbb{Z}$ and a is a constant
- The **alternative notation** (to $\mathrm{f}'(x)$) is to use $\frac{\mathrm{d}y}{\mathrm{d}x}$
  - If $y = ax^n$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = anx^{n-1}$
    - e.g. If $y = -4x^5$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = -4 \times 5x^{5-1} = -20x^4$
- Don't forget these **two** special cases:
  - If $\mathrm{f}(x) = ax$ then $\mathrm{f}'(x) = a$
    - e.g. If $y = 6x$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = 6$
  - If $\mathrm{f}(x) = a$ then $\mathrm{f}'(x) = 0$
    - e.g. If $y = 5$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$
  - These allow you to differentiate **linear terms** in x and **constants**
- Functions involving **fractions** with **denominators** in terms of x will need to be rewritten as **negative powers** of x first
  - If $\mathrm{f}(x) = \frac{4}{x}$ then rewrite as $\mathrm{f}(x) = 4x^{-1}$ and differentiate

How do I differentiate sums and differences of powers of x?

- The formulae for differentiating powers of x apply to **all integer** powers so it is possible to differentiate any expression that is a **sum** or **difference** of **powers** of x
  - e.g. If $\mathrm{f}(x) = 5x^4 + 2x^3 - 3x + 4$ then
    $\mathrm{f}'(x) = 5 \times 4x^{4-1} + 2 \times 3x^{3-1} - 3 + 0$
    $\mathrm{f}'(x) = 20x^3 + 6x^2 - 3$
- **Products** and **quotients cannot** be differentiated in this way so would need **expanding/simplifying** first
  - e.g. If $\mathrm{f}(x) = (2x-3)(x^2-4)$ then expand to $\mathrm{f}(x) = 2x^3 - 3x^2 - 8x + 12$ which **is** a **sum/difference** of powers of x and can be differentiated

Exam Tip

- A common mistake is not simplifying expressions before differentiating
    - The derivative of $(x^2+3)(x^3-2x+1)$ can **not** be found by multiplying the derivatives of (x^2+3) and (x^3-2x+1)

Worked Example

The function $\mathrm{f}(x)$ is given by

$$\mathrm{f}(x) = x^3 - 2x^2 + 3 - \frac{4}{x^3}$$

a)

Find the derivative of $\mathrm{f}(x)$.

Rewrite f(x) so every term is a power of x

$f(x) = x^3 - 2x^2 + 3 - 4x^{-3}$

Differentiate by applying the formula (3 is a special case)

$f'(x) = 3x^2 - 4x + 12x^{-4}$

nx^{n-1} ↗ anx^{n-1} ↑ ↖ take care with negatives

$$\boxed{\therefore\ f'(x) = 3x^2 - 4x + \frac{12}{x^4}}$$

b)

Find the gradient of the tangent to the curve $y = \mathrm{f}(x)$ at the points where $x = -1$ and $x = 1$.

$f'(-1) = 3(-1)^2 - 4(-1) + \frac{12}{(-1)^4} = 3 + 4 + 12 = 19$

$f'(1) = 3(1)^2 - 4(1) + \frac{12}{(1)^4} = 3 - 4 + 12 = 11$

∴ The gradient of the tangent to the curve $y = f(x)$ when $x = -1$ is 19, and when $x = 1$, is 11

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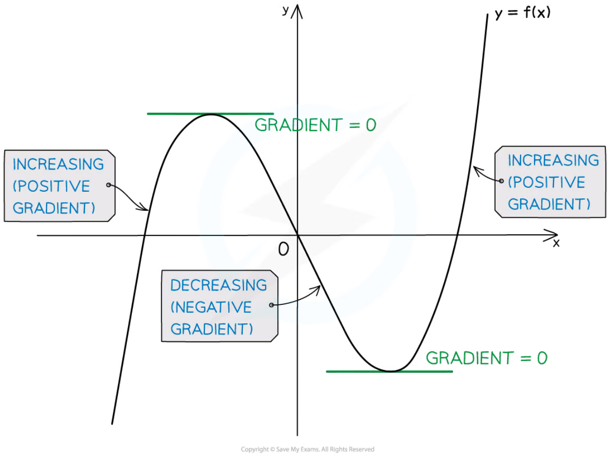
Increasing & Decreasing Functions

What are increasing and decreasing functions?

- A function, f(x), is **increasing** if **f'(*x*) > 0**
    - This means the value of the function ('output') increases as x increases
- A function, f(x), is **decreasing** if **f'(*x*) < 0**
    - This means the value of the function ('output') decreases as x increases
- A function, f(x), is **stationary** if **f'(*x*) = 0**



How do I find where functions are increasing, decreasing or stationary?

- To identify the **intervals** on which a function is increasing or decreasing

STEP 1 Find the derivative **f'(*x*)**
STEP 2 Solve the inequalities **f'(*x*) > 0** (for increasing intervals) and/or **f'(*x*) < 0** (for decreasing intervals)

- Most functions are a combination of increasing, decreasing and stationary
    - a range of values of x (**interval**) is given where a function satisfies each condition
    - e.g. The function $\mathrm{f}(x) = x^2$ has **derivative** $\mathrm{f}'(x) = 2x$ so
        - $\mathrm{f}(x)$ is **decreasing** for $x < 0$
        - $\mathrm{f}(x)$ is **stationary** at $x = 0$
        - $\mathrm{f}(x)$ is **increasing** for $x > 0$

Worked Example

$f(x) = x^2 - x - 2$

a)
Determine whether $f(x)$ is increasing or decreasing at the points where $x = 0$ and $x = 3$.

Differentiate

$f'(x) = 2x - 1$

At $x = 0$, $f'(0) = 2 \times 0 - 1 = -1 < 0$ ∴ decreasing

At $x = 3$, $f'(3) = 2 \times 3 - 1 = 5 > 0$ ∴ increasing

∴ At $x = 0$, $f(x)$ is decreasing
At $x = 3$, $f(x)$ is increasing

b)
Find the values of x for which $f(x)$ is an increasing function.

$f(x)$ is increasing when $f'(x) > 0$

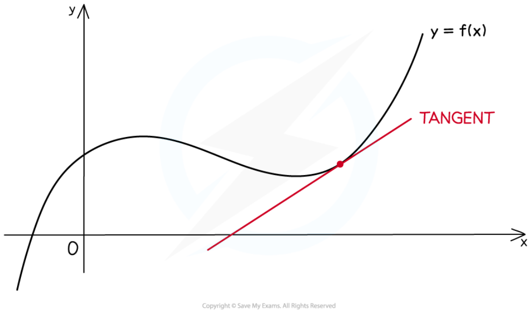
$f'(x) > 0$
$2x - 1 > 0$
$x > \frac{1}{2}$

∴ $f(x)$ is increasing for $x > \frac{1}{2}$

Tangents & Normals

What is a tangent?

- At any point on the graph of a (non-linear) **function**, the **tangent** is the straight line that passes through that point and has the same **gradient** as the curve at that point



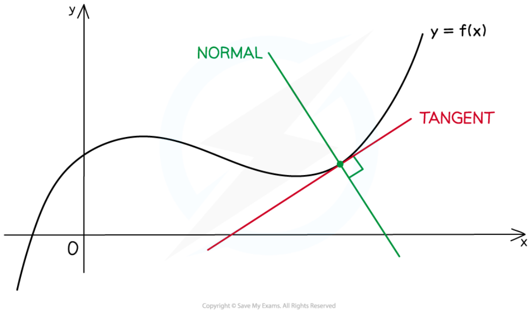
How do I find the equation of a tangent?

- The **equation** of the **tangent** to the function $y = \mathrm{f}(x)$ at the point (x_1, y_1) is

$$\boldsymbol{y - y_1 = f'(x_1)(x - x_1)}$$

What is a normal?

- At any point on the graph of a (non-linear) function, the **normal** is the straight line that passes through and is **perpendicular** to the **tangent** at that point



How do I find the equation of a normal?

- The **equation** of the **normal** to the function $y = \mathrm{f}(x)$ at the point (x_1, y_1) is

$$\boldsymbol{y - y_1 = \frac{-1}{f'(x_1)}(x - x_1)}$$

Exam Tip

- You are not given the formula for the equation of a tangent and equation of a normal
- Both can be derived from the equation of a straight line $y - y_1 = m(x - x_1)$ which is given

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Worked Example

The function $f(x)$ is defined by

$$f(x) = 2x^4 + \frac{3}{x^2} \qquad x \neq 0$$

a)
Find an equation for the tangent to the curve $y = f(x)$ at the point where $x = 1$, giving your answer in the form $y = mx + c$.

First find $f'(x)$ by differentiating

$f(x) = 2x^4 + 3x^{-2}$ Rewrite as powers of x

$f'(x) = 8x^3 - 6x^{-3}$

For a tangent, "$y - y_1 = f'(a)(x - x_1)$"

At $x = 1$, $y = 2(1)^4 + \frac{3}{(1)^2} = 5$

$f'(1) = 8(1)^3 - \frac{6}{(1)^3} = 2$

$\therefore$ $y - 5 = 2(x - 1)$

$\boxed{\text{Tangent at } x = 1, \text{ is } y = 2x + 3}$

b)
Find an equation for the normal at the point where $x = 1$, giving your answer in the form $ax + by + d = 0$, where a, b and d are integers.

For a normal, "$y - y_1 = \frac{-1}{f'(a)}(x - x_1)$"

Using results from part (a):

$y - 5 = \frac{-1}{2}(x - 1)$

$y = -\frac{1}{2}x + \frac{11}{2}$

$2y = -x + 11$

$\boxed{\therefore \text{Equation of normal is } x + 2y - 11 = 0}$

Local Minimum & Maximum Points

What are local minimum and maximum points?

- Local **minimum** and **maximum** points are two types of **stationary** point
  - The **gradient function** (derivative) at such points equals zero
    i.e. $f'(x)=0$
- A **local minimum** point, $(x,\ f(x))$ will be the **lowest** value of $f(x)$ in the **local** vicinity of the value of x
  - The function may reach a **lower** value further afield
- Similarly, a **local maximum** point, $(x,\ f(x))$ will be the **greatest** value of $f(x)$ in the **local** vicinity of the value of x
  - The function may reach a **greater** value further afield
- The graphs of many functions tend to infinity for large values of x
  (and/or minus infinity for large negative values of x)
- The **nature** of a stationary point refers to whether it is a local **minimum** or local **maximum** point

How do I find the coordinates and nature of stationary points?

- The instructions below describe how to find **local minimum** and **maximum points** using a **GDC** on the graph of the function $y=f(x)$.

STEP 1
Plot the graph of $y=f(x)$
Sketch the graph as part of the solution

STEP 2
Use the options from the graphing screen to "solve for minimum"
The GDC will display the x and y coordinates of the first minimum point
Scroll onwards to see there are anymore minimum points
Note down the coordinates and the type of stationary point

STEP 3
Repeat **STEP 2** but use "solve for maximum" on your GDC

- In **STEP 2** the **nature** of the stationary point should be easy to tell from the graph
  - a local **minimum** changes the function from **decreasing** to **increasing**
    - the gradient changes from **negative** to **positive**
  - a local **maximum** changes the function from **increasing** to **decreasing**
    - the gradient changes from **positive** to **negative**

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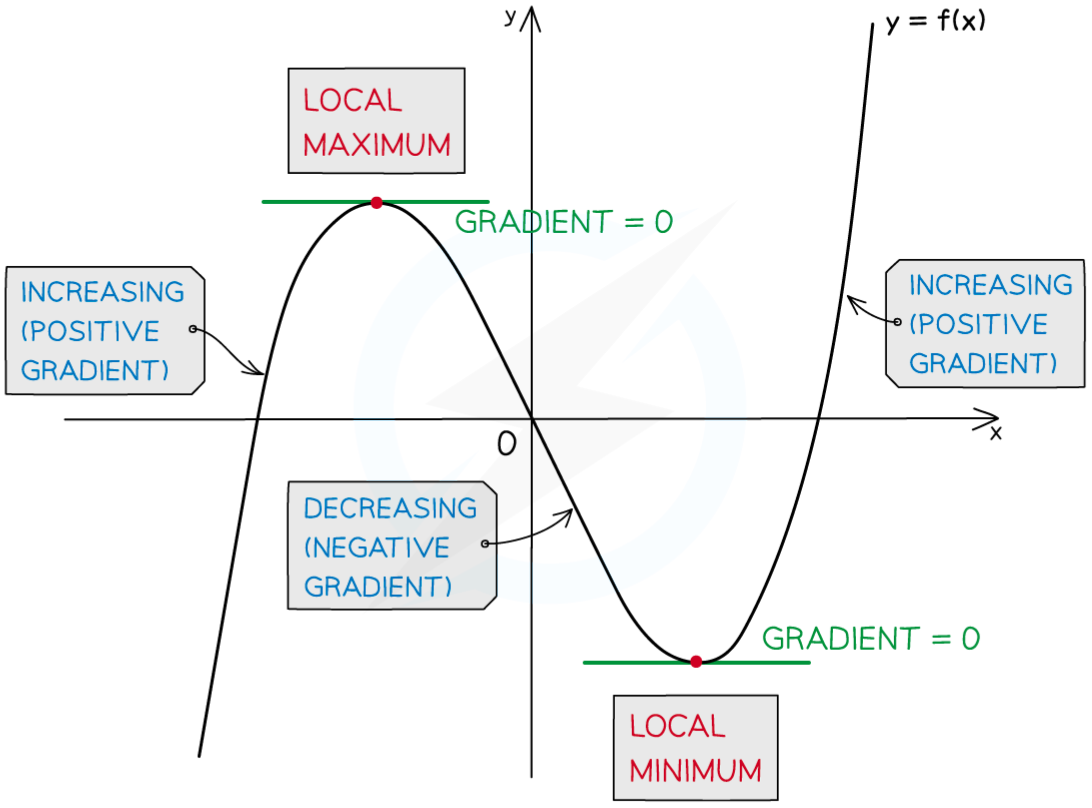
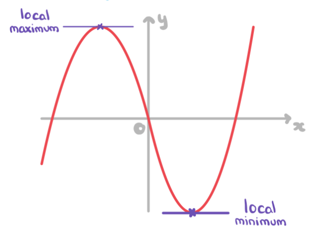


Worked Example

Find the stationary points of $f(x) = x(x^2 - 27)$, and state their nature.

Plot the graph of $y = x(x^2-27)$ on GDC and sketch here



∴ Stationary points are

(3, -54) LOCAL MINIMUM POINT
(-3, 54) LOCAL MAXIMUM POINT

5.1.3 Modelling with Differentiation

Modelling with Differentiation

What can be modelled with differentiation?

- Recall that **differentiation** is about the **rate** of **change** of a function and provides a way of finding **minimum** and **maximum** values of a function
- Anything that involves **maximising** or **minimising** a quantity can be modelled using differentiation; for example
  - **minimising** the cost of raw materials in manufacturing a product
  - the **maximum** height a football could reach when kicked
- These are called **optimisation** problems

What modelling assumptions are used in optimisation problems?

- The quantity being **optimised** needs to be dependent on a **single** variable
  - If other variables are initially involved, **constraints** or **assumptions** about them will need to be made; for example
    - minimising the cost of the **main** raw material – timber in manufacturing furniture say – the cost of screws, glue, varnish, etc can be fixed or considered **negligible**
  - Other **modelling assumptions** may have to be made too; for example
    - ignoring air resistance and wind when modelling the path of a kicked football

How do I solve optimisation problems?

- In optimisation problems, letters other than x, y and f are often used including capital letters
  - V is often used for volume, S for surface area
  - r for radius if a circle, cylinder or sphere is involved
- **Derivatives** can still be found but be clear about which variable is independent (x) and which is dependent (y)
  - a GDC may always use x and y but ensure you use the correct variable throughout your working and final answer
- Problems often start by **linking two connected** quantities together – for example **volume** and **surface area**
  - where more than one variable is involved, **constraints** will be given such that the quantity of interest can be rewritten in terms of a **single** variable
- Once the quantity of interest is written as a function of a **single** variable, **differentiation** can be used to **maximise** or **minimise** the quantity as required

STEP 1

Rewrite the quantity to be optimised as a single variable, using any constraints given in the question

STEP 2

Use your GDC to find the (local) maximum or minimum points as required
Plot the graph of the function and use the graphing features of the GDC to "solve for minimum/maximum" as required

STEP 3
Note down the solution from your GDC and interpret the answer(s) in the context of the question

Exam Tip

- The first part of rewriting a quantity as a single variable is often a “show that” question – this means you may still be able to access later parts of the question even if you can’t do this bit

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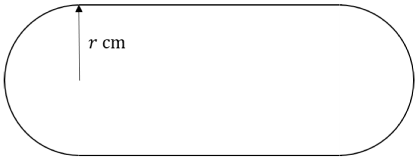
Worked Example

A large allotment bed is being designed as a rectangle with a semicircle on each end, as shown in the diagram below.



The total area of the bed is to be $100\pi \text{ m}^2$.

a)
Show that the perimeter of the bed is given by the formula

$$P = \pi\left(r + \frac{100}{r}\right)$$

The width of the rectangle is $2r$ m and its length L m

The AREA of the bed, 100π m^2 is given by

$$\underbrace{\tfrac{1}{2}\pi r^2}_{\text{semi-circle}} + \underbrace{2rL}_{\text{rectangle}} + \underbrace{\tfrac{1}{2}\pi r^2}_{\text{semi-circle}} = \underbrace{100\pi}_{\text{total area (this is the constraint)}}$$

$$\therefore \pi r^2 + 2rL = 100\pi$$

$$2rL = 100\pi - \pi r^2$$

$$L = \frac{50\pi}{r} - \frac{\pi}{2}r$$

Write L in terms of r

The PERIMETER of the bed is

$$P = \underbrace{\pi r + \pi r}_{\text{semi-circular arcs}} + \underbrace{2L}_{\text{two straight lengths}}$$

Use L from the area constraint to write P in terms of r only

$$P = 2\pi r + 2\left(\frac{50\pi}{r} - \frac{\pi}{2}r\right)$$

$$P = \pi r + \frac{100\pi}{r}$$

$$\boxed{\therefore P = \pi\left(r + \frac{100}{r}\right)}$$

b) Find $\frac{dP}{dr}$.

Rewrite P as powers of r

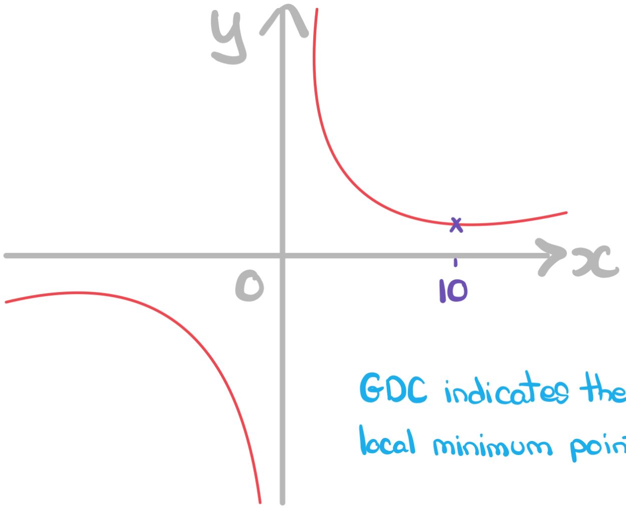
$P = \pi(r + 100r^{-1})$

$\frac{dP}{dr} = \pi(1 - 100r^{-2})$

$$\boxed{\therefore \frac{dP}{dr} = \pi\left(1 - \frac{100}{r^2}\right)}$$

c)
Find the value of r that minimises the perimeter.

Use GDC to plot $y = \pi\left(x + \frac{100}{x}\right)$ and sketch the result



GDC indicates the ONLY local minimum point is at $x = 10$

$$\boxed{\therefore \text{The value of } r \text{ that minimises the perimeter is } r = 10}$$

d)
Hence find the minimum perimeter.

The minimum perimeter will be the y-coordinate of the local minimum point found in part (c)

From GDC, $y = 62.831\,853...$ (when $x = 10$)

$\therefore$ **Minimum perimeter is 62.8 m (3 s.f.)**

5.2 Integration

5.2.1 Trapezoid Rule: Numerical Integration

Trapezoid Rule: Numerical Integration

What is the trapezoid rule?

- The **trapezoidal rule** is a numerical method used to find the **approximate area** enclosed by a curve, the x-axis and two vertical lines
  - it is also known as '**trapezoid rule**' and '**trapezium rule**'
- The trapezoidal rule finds an **approximation** of the area by **summing of the areas** of trapezoids beneath the curve
  - $y_0 = f(a),\ \ y_1 = f(a+h),\ \ y_2 = f(a+2h)$ etc

$$\int_a^b f(x)\ dx \approx \frac{1}{2}h\left[\left(y_0 + y_n\right) + 2\left(y_1 + y_2 + \ldots + y_{n-1}\right)\right]$$

$$\text{where } h = \frac{b-a}{n}$$

- Note that there are n trapezoids (also called strips) but $(n+1)$ function values (y_i)
- The trapezoidal rule is given in the **formula booklet**

What else can be asked to do with the trapezoid rule?

- Comparing the **true** answer with the answer from the trapezoid rule
  - This may involve finding the **percentage error** in the approximation
  - The true answer may be given in the question, found from a GDC or from work on **integration**

Exam Tip

- Ensure you are clear about the difference between the number of data points (y values) and the number of strips (number of trapezoids) used in a Trapezoid Rule question
- Although it shouldn't be too much trouble to type the trapezoid rule into your GDC in one go, it may be wise to work parts of it out separately and write these down as part of your working out

Worked Example

a)

Using the trapezoidal rule, find an approximate value for

$$\int_0^4 \frac{6x^2}{x^3+2}\,dx$$

to 3 decimal places, using $n = 4$.

a) $h = \frac{4-0}{4} = 1$

STEP 1: FIND $h = \frac{b-a}{n}$ USING $\int_{a=0}^{b=4}$ AND $n = 4$

STEP 2: USING WIDTH $h = 1$, VALUES ARE 0, 1, 2, 3, 4

STEP 3: SETUP TABLE AND USE EQUATION TO CALCULATE y VALUES

x	$y = \frac{6x^2}{x^3+2}$
$x_0 = 0$	$y_0 = 0$
$x_1 = 1$	$y_1 = 2$
$x_2 = 2$	$y_2 = 2.4$
$x_3 = 3$	$y_3 = 1.862...$
$x_4 = 4$	$y_4 = 1.454...$

STEP 4: USING FORMULA FROM FORMULA BOOKLET

$$\int_a^b y\,dx \approx \frac{1}{2}h(y_0 + y_n + 2(y_1 + y_2 + ... + y_{n-1}))$$

SUBSTITUTE ALL VALUES

$$\frac{1}{2} \times 1 \times (0 + 1.454... + 2(2 + 2.4 + 1.862...))$$

$$= \frac{1}{2}(1.454... + 12.524...)$$

$$= 6.9893... = 6.989 \quad (3\text{ dp})$$

b)

Given that the area bounded by the curve , the x-axis and the lines $x = 0$ and $x = 4$ is 6.993 to three decimal places, calculate the percentage error in the trapezoidal rule approximation.

USING $\%\text{ ERROR} = \frac{\text{ESTIMATE} - \text{EXACT}}{\text{EXACT}} \times 100$

b) $$\frac{6.989 - 6.993}{6.993} \times 100$$

$$= -0.\dot{0}5720\dot{0}$$

IGNORE MINUS, YOU COULD DO EXACT − ESTIMATE TO GET RID OFF THIS

$$= 0.06\% \quad (2\text{ dp})$$

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Introduction to Integration

What is integration?

- **Integration** is the opposite to **differentiation**
    - Integration is referred to as **antidifferentiation**
    - The result of integration is referred to as the **antiderivative**
- **Integration** is the process of finding the expression of a function (**antiderivative**) from an expression of the **derivative** (**gradient function**)

What is the notation for integration?

- An **integral** is normally written in the form

$$\int \mathrm{f}(x)\,\mathrm{d}x$$

    - the large operator $\int$ means "integrate"
    - "$\mathrm{d}x$" indicates which variable to integrate with respect to
    - $\mathrm{f}(x)$ is the function to be integrated (sometimes called the integrand)
- The **antiderivative** is sometimes denoted by $\mathrm{F}(x)$
    - there's then no need to keep writing the whole integral; refer to it as $\mathrm{F}(x)$
- $\mathrm{F}(x)$ may also be called the **indefinite integral** of $\mathrm{f}(x)$

What is the constant of integration?

- Recall one of the special cases from **Differentiating Powers of x**
    - If $\mathrm{f}(x) = a$ then $\mathrm{f}'(x) = 0$
- This means that integrating 0 will produce a **constant** term in the antiderivative
    - a zero term wouldn't be written as part of a function
    - **every** function, when integrated, potentially has a **constant** term
- This is called the **constant** of **integration** and is usually denoted by the letter c
    - it is often referred to as "plus c"
- Without more information it is impossible to deduce the value of this constant
    - there are endless antiderivatives, $\mathrm{F}(x)$, for a function $\mathrm{f}(x)$

Integrating Powers of x

How do I integrate powers of x?

- Powers of x are integrated according to the following formula:
  - If $\mathrm{f}(x) = x^n$ then $\int \mathrm{f}(x)\,\mathrm{d}x = \frac{x^{n+1}}{n+1} + c$ where $n \in \mathbb{Z},\quad n \neq -1$ and c is the **constant** of **integration**
- This is given in the **formula booklet**
- If the power of is x multiplied by a constant then the integral is also multiplied by that constant
  - If $\mathrm{f}(x) = ax^n$ then $\int \mathrm{f}(x)\,\mathrm{d}x = \frac{ax^{n+1}}{n+1} + c$ where $n \in \mathbb{Z},\quad n \neq -1,\ a$ is a constant and c is the **constant** of **integration**
- This is also given in the **formula booklet**
- $\frac{\mathrm{d}y}{\mathrm{d}x}$ notation can still be used with integration
- Note that the formulae above do not apply when $n = -1$ as this would lead to division by zero
- Don't forget the special case:
  - $\int a\,\mathrm{d}x = ax + c$
    - e.g. $\int 4\,\mathrm{d}x = 4x + c$
  - This allows **constant** terms to be integrated
- Functions involving **fractions** with **denominators** in terms of x will need to be rewritten as **negative powers** of x first
  - e.g. If $\mathrm{f}(x) = \frac{4}{x^2}$ then rewrite as $\mathrm{f}(x) = 4x^{-2}$ and integrate

How do I integrate sums and differences of powers of x?

- The formulae for integrating powers of x apply to **all integers** so it is possible to integrate any expression that is a **sum** or **difference** of powers of x
  - e.g. If $\mathrm{f}(x) = 8x^3 - 2x + 4$ then
    $$\int \mathrm{f}(x)\,\mathrm{d}x = \frac{8x^{3+1}}{3+1} - \frac{2x^{1+1}}{1+1} + 4x + c = 2x^4 - x^2 + 4x + c$$
- **Products** and **quotients** cannot be integrated in this way so would need **expanding/simplifying** first
  - e.g. If $\mathrm{f}(x) = 8x^2(2x - 3)$ then
    $$\int \mathrm{f}(x)\,\mathrm{d}x = \int (16x^3 - 24x^2)\,\mathrm{d}x = \frac{16x^4}{4} - \frac{24x^3}{3} + c = 4x^4 - 8x^3 + c$$

Exam Tip

- You can speed up the process of integration in the exam by committing the pattern of basic integration to memory
    - In general you can think of it as 'raising the power by one and dividing by the new power'
    - Practice this lots before your exam so that it comes quickly and naturally when doing more complicated integration questions

Worked Example

Given that

$$\frac{dy}{dx} = 3x^4 - 2x^2 + 3 - \frac{1}{x^4}$$

find an expression for y in terms of x.

Firstly rewrite all terms as powers of x

$$\frac{dy}{dx} = 3x^4 - 2x^2 + 3 - x^{-4}$$

$$y = \int \left(3x^4 - 2x^2 + 3 - x^{-4}\right) dx$$

$$\therefore y = \frac{3x^5}{5} - \frac{2x^3}{3} + 3x - \frac{x^{-3}}{-3} + c$$

special case

take care with negatives, $-4+1=-3$

constant of integration

$$\boxed{\therefore y = \frac{3}{5}x^5 - \frac{2}{3}x^3 + 3x + \frac{1}{3x^3} + c}$$

Finding the Constant of Integration

What is the constant of integration?

- When finding an **anti-derivative** there is a constant term to consider
  - this constant term, usually called c, is the **constant** of **integration**
- In terms of **graphing** an **anti-derivative**, there are endless possibilities
  - collectively these may be referred to as the **family** of **antiderivatives** or **family** of **curves**
  - the constant of integration is determined by the **exact** location of the curve
    - if a **point** on the **curve** is **known,** the **constant** of **integration** can be found

How do I find the constant of integration?

- For $\mathrm{F}(x) + c = \int f(x)\,\mathrm{d}x$, the **constant** of **integration,** c – and so the particular **antiderivative** - can be found if a point the graph of $y = \mathrm{F}(x) + c$ passes through is known

STEP 1
If need be, rewrite $f(x)$ into an integrable form
Each term needs to be a power of x (or a constant)

STEP 2
Integrate each term of $f'(x)$, remembering the constant of integration, "$+c$"
(Increase power by 1 and divide by new power)

STEP 3
Substitute the x and y coordinates of a given point in to $\mathrm{F}(x) + c$ to form an equation in c
Solve the equation to find c

Exam Tip

- If a constant of integration can be found then the question will need to give you some extra information
  - If this is given then make sure you use it to find the value of c

YOUR NOTES
⬇

Worked Example

The graph of $y = f(x)$ passes through the point $(3, -4)$. The gradient function of $f(x)$ is given by $f'(x) = 3x^2 - 4x - 4$.

Find $f(x)$.

STEP 1 $f'(x)$ is already in an integrable form

$f'(x) = 3x^2 - 4x - 4$

STEP 2 Integrate, remembering "+c"

$$f(x) = \frac{3x^3}{3} - \frac{4x^2}{2} - 4x + c$$

$$f(x) = x^3 - 2x^2 - 4x + c$$

STEP 3 Substitute x and y coordinates to find c

$f(3) = -4$

$\therefore (3)^3 - 2(3)^2 - 4(3) + c = -4$

$27 - 18 - 12 + c = -4$

$c = -1$

$$\boxed{\therefore f(x) = x^3 - 2x^2 - 4x - 1}$$

YOUR NOTES
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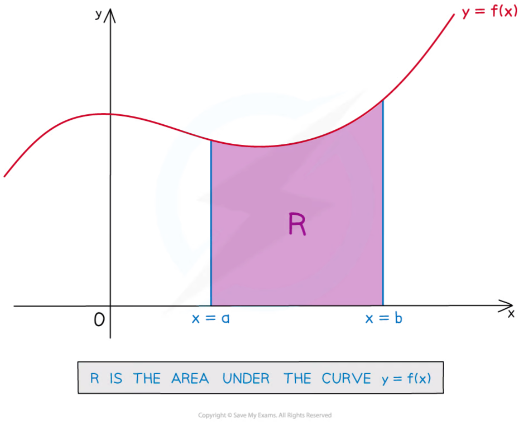
Area Under a Curve Basics

What is meant by the area under a curve?



- The phrase "**area under a curve"** refers to the area bounded by
  - the graph of $y = f(x)$
  - the x-axis
  - the **vertical** line $x = a$
  - the **vertical** line $x = b$
- The **exact area under a curve** is found by evaluating a **definite integral**
- The graph of $y = f(x)$ could be a **straight line**
  - the use of **integration** described below would still apply
    - but the shape created would be a **trapezoid**
    - so it is easier to use "$A = \frac{1}{2}h(a+b)$"

What is a definite integral?

$$\int_a^b f(x)\,\mathrm{d}x = \mathrm{F}(a) - \mathrm{F}(b)$$

- This is known as the **Fundamental Theorem of Calculus**
- **a** and **b** are called limits
  - **a** is the **lower** limit
  - **b** is the **upper** limit
- $f(x)$ is the **integrand**
- $\mathrm{F}(x)$ is an **antiderivative** of $f(x)$
- The **constant** of **integration** ("$+c$") is not needed in **definite integration**
  - "$+c$" would appear alongside both **F(a)** and **F(b)**
  - subtracting means the "$+c$"'s cancel

How do I form a definite integral to find the area under a curve?

- The graph of $y = f(x)$ and the x-axis should be obvious boundaries for the area so the key here is in finding a and b - the **lower** and **upper** limits of the **integral**

STEP 1

Use the given sketch to help locate the limits
You may prefer to plot the graph on your GDC and find the limits from there

STEP 2
Look carefully where the 'left' and 'right' boundaries of the area lie
If the boundaries are vertical lines, the limits will come directly from their equations
Look out for the y-axis being one of the (vertical) boudnaries – in this case the limit (x) will be 0
One, or both, of the limits, could be a root of the equation $f(x) = 0$
i.e. where the graph of $y = f(x)$ crosses the x-axis
In this case solve the equation $f(x) = 0$ to find the limit(s)
A GDC will solve this equation, either from the graphing screen or the equation solver

STEP 3
The definite integral for finding the area can now be set up in the form

$$A = \int_a^b f(x)\ \mathrm{d}x$$

Exam Tip

- Look out for questions that ask you to find an **indefinite** integral in one part (so "**+c**" needed), then in a later part use the same integral as a **definite** integral (where "**+c**" is not needed)
- Add information to any diagram provided in the question, as well as axes intercepts and values of limits
  - Mark and shade the area you're trying to find, and if no diagram is provided, **sketch** one!

Definite Integrals using GDC

Does my calculator/GDC do definite integrals?

- Modern graphic calculators (and some 'advanced' scientific calculators) have the functionality to evaluate **definite integrals**
  - i.e. they can calculate the **area under a curve** (see above)
- If a calculator has a button for evalutaing definite integrals it will look something like

$$\int_{\square}^{\square} \square$$

- This may be a physical button or accessed via an on-screen menu
- Some GDCs may have the ability to find the area under a curve from the graphing screen
- Be careful with **any** calculator/GDC, they may not produce an **exact** answer

How do I use my GDC to find definite integrals?

Without graphing first ...

- Once you know the **definite integral** function your calculator will need three things in order to evaluate it
  - The function to be integrated (**integrand**) ($f(x)$)
  - The **lower** limit (a from $x = a$)
  - The **upper** limit (b from $x = b$)
- Have a play with the order in which your calculator expects these to be entered – some do not always work left to right as it appears on screen!

With graphing first ...

- Plot the graph of $y = f(x)$
  - You may also wish to plot the vertical lines $x = a$ and $x = b$
    - make sure your GDC is expecting an "$x =$ " style equation
  - Once you have plotted the graph you need to look for an option regarding "area" or a physical button
    - it may appear as the integral symbol (e.g. $\int \mathrm{d}x$)
    - your GDC may allow you to select the lower and upper limits by moving a cursor along the curve - however this may not be very accurate
    - your GDC may allow you to type the exact limits required from the keypad
      - the lower limit would be typed in first
      - read any information that appears on screen carefully to make sure

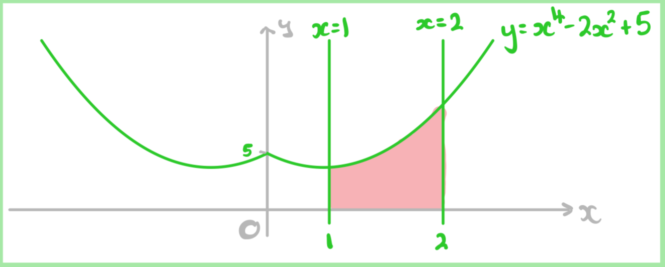
Exam Tip

- When revising for your exams always use your GDC to check any definite integrals you have carried out by hand
  - This will ensure you are confident using the calculator you plan to take into the exam and should also get you into the habit of using you GDC to check your work, something you should do if possible

Worked Example

a)
Using your GDC to help, or otherwise, sketch the graphs of
$y = x^4 - 2x^2 + 5$,
$x = 1$ and
$x = 2$ on the same diagram

Use the 'graph' menu on your GDC to plot $y = x^4 - 2x^2 + 5$.
You may then need to change the 'input type' to '$x =$'
to enter $x = 1$ and $x = 2$.
Plot the graph on your GDC and sketch the result, ensuring
to include all the main properties of each graph.



b)
The area enclosed by the three graphs from part (a) and the x-axis is to be found.
Write down an integral that would find this area.

$$\int_1^2 (x^4 - 2x^2 + 5)\, dx$$

c)
Using your GDC, or otherwise, find the exact area described in part (b).

Give your answer in the form $\frac{a}{b}$ where a and b are integers.

$$\text{Area} = \int_1^2 (x^4 - 2x^2 + 5)\, dx = \frac{98}{15} \text{ square units}$$

From the graphing screen on our GDC the integral
value was given as 6·53333333 – not exact!