5.9 Advanced Integration

Question Paper

Course	DP IB Maths
Section	5. Calculus
Topic	5.9 Advanced Integration
Difficulty	Medium

Time allowed: 130

Score: /101

Percentage: /100

Question 1a

Find the following indefinite integrals:

(a)

$\int 4\sec^2 2x \, \mathrm{d}x$

[2 marks]

Question 1b

(b)

$\int \sec \frac{x}{3} \tan \frac{x}{3} \, \mathrm{d}x$

[2 marks]

Question 1c

(c)

$\int \frac{1}{\sin^2\left(x + \frac{\pi}{4}\right)} \, \mathrm{d}x$

[3 marks]

Question 2a

Find the following indefinite integrals:

(a)

$\int (\ln 3) 3^x \, \mathrm{d}x$

[2 marks]

Question 2b

(b)

$\int \frac{12}{9 + x^2} \, \mathrm{d}x$

[2 marks]

Question 2c

(c)

$\int \frac{3}{5\sqrt{16 - x^2}} \mathrm{d}x$

[2 marks]

Question 2d

(d)

Using a sketch, briefly describe the family of graphs corresponding to all the possible specific solutions to the integral in part (a).

[3 marks]

Question 3a

(a)
Show that

$$\frac{5}{x^2+9x+14} = \frac{A}{x+2} + \frac{B}{x+7}$$

where A and B are constants to be found.

[2 marks]

Question 3b

(b)
Hence, find the indefinite integral

$$\int \frac{5}{x^2+9x+14}\,\mathrm{d}x$$

using laws of logarithms to simplify your answer as far as possible.

[4 marks]

Question 3c

(c)
Show that

$$\int_{-1}^{5} \frac{5}{x^2+9x+14}\,\mathrm{d}x = \ln 7 - \ln 2.$$

[4 marks]

Question 4a

(a)
Use the substitution $u = x - 3$ to find the following indefinite integral:

$$\int x\sqrt{x-3}\,\mathrm{d}x.$$

[4 marks]

Question 4b

(b)
Hence find the value of the definite integral

$$\int_4^7 x\sqrt{x-3}\,dx$$

(i)
by evaluating the definite integral entirely in terms of x

(ii)
by converting the integral limits to appropriate values of u and evaluating the definite integral entirely in terms of u.

Verify that the two methods give the same result for the value of the integral.

[4 marks]

Question 5a

(a)
Show that $x^2 - 10x + 29$ may be written in the form $p + (x - q)^2$, where p and q are constants to be determined.

[2 marks]

Question 5b

(b)

Using your results from part (a) along with the substitution $u = x - q$, show that

$$\int \frac{1}{x^2 - 10x + 29}\, \mathrm{d}x = \frac{1}{2}\arctan\left(\frac{x-5}{2}\right) + c\,.$$

[5 marks]

Question 5c

(c) Find the exact value of the definite integral

$$\int_5^7 \frac{1}{x^2 - 10x + 29}\, dx$$

[3 marks]

Question 6

Use the substitution $u = 1 + \sin^3 x$ to show that

$$\int_0^{\frac{\pi}{2}} \sin^2 x \cos x \sqrt{1 + \sin^3 x}\ \mathrm{d}x = \frac{2}{9}\left(2\sqrt{2} - 1\right).$$

[6 marks]

Question 7a

(a)

Use integration by parts to find the indefinite integral

$$\int x\mathrm{e}^{2x}\ \mathrm{d}x.$$

[4 marks]

Question 7b

(b)
Hence find the exact value of the definite integral

$$\int_0^3 x\mathrm{e}^{2x}\,\mathrm{d}x.$$

[3 marks]

Question 7c

(c)
Use technology to evaluate the integral in part (b), and compare this to the exact value you found.

[1 mark]

Question 8a

(a)
Use integration by parts twice to show that

$$\int 32x^2\mathrm{e}^{4x}\,\mathrm{d}x = \mathrm{e}^{4x}(px^2 + qx + r) + c$$

where p, q and r are constants to be found, and where c is a constant of integration.

[6 marks]

Question 8b

Let f be a function defined for all $x \in \mathbb{R}$. Consider the graph of $y = f(x)$.

(b)
Given that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 32x^2\mathrm{e}^{4x}$$

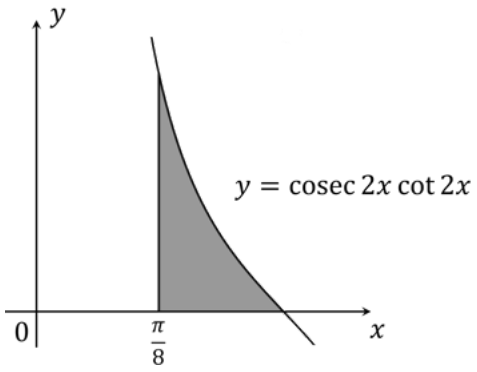
and that the graph passes through the point $\left(\frac{1}{4}, \frac{\mathrm{e}+7}{2}\right)$, find an expression for $f(x)$.

[3 marks]

Question 9

Let f be the function defined by $f(x) = \text{cosec } 2x \cot 2x,\ 0 < x < \frac{\pi}{2}$.

The diagram below shows a part of the graph of the curve $y = f(x)$. The shaded region is the region bounded by the curve, the positive x-axis and the line $x = \frac{\pi}{8}$.

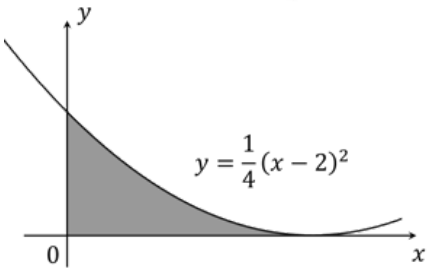


Find the exact area of the shaded region.

[6 marks]

Question 10a

The following diagram shows a part of the graph of the curve $y = \frac{1}{4}(x-2)^2$. The shaded region is the region enclosed by the graph and the positive x- and y-axes.



(a)

(i)

Find the coordinates of the points where the graph intersects the coordinate axes.

(ii)

For the part of the curve that forms the boundary of the shaded region, show that $x = 2 - 2\sqrt{y}$.

[3 marks]

Question 10b

(b)

Find the area of the shaded region

(i)

by calculating it as an area between the curve and the x-axis.

(ii)

by calculating it as an area between the curve and the y-axis.

[6 marks]

Question 10c

(c)
Find the volume of the solid formed when the shaded region is rotated 2π radians about the x-axis.

[5 marks]

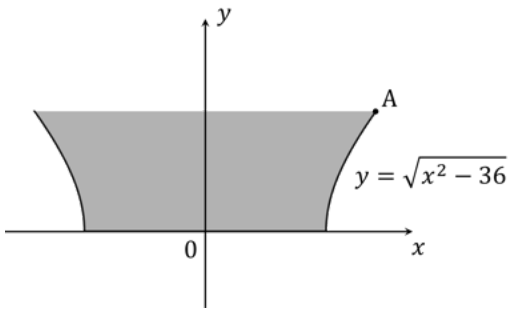
Question 10d

(d)
Find the volume of the solid formed when the shaded region is rotated 2π radians about the y-axis.

[5 marks]

Question 11a

The diagram below shows the cross-section of a bowl that a company is planning to begin producing.



As indicated on the diagram, one of the sides of the bowl in the cross-section may be described by the curve $y = \sqrt{x^2 - 36}$, where units for x and y are centimetres. The cross-section is entirely symmetrical about the y-axis. The flat circular bottom of the bowl has a diameter of 12 cm, and the vertical depth of the bowl is 6 cm. For purposes of answering this question, the thickness of the bottom and sides of the bowl may be regarded as negligible.

(a)
Find the exact coordinates of the point marked A on the diagram.

[3 marks]

Question 11b

(b)
Show that the capacity of the bowl in cm^3 is given by

$$\pi \int_0^b (y^2 + 36) \mathrm{d}y$$

where b is a constant to be determined.

[4 marks]

Question 11c

(c)
Hence find the capacity of the bowl.

[2 marks]