5.6 Differential Equations

Question Paper

Course	DP IB Maths
Section	5. Calculus
Topic	5.6 Differential Equations
Difficulty	Very Hard

Time allowed: 90

Score: /67

Percentage: /100

Question 1

Consider the first-order differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{2x} = \sin 3x \cos 3x$$

By first finding the general solution to the equation, solve the equation for the case that y=0 when $x = \frac{\pi}{2}$.

[5 marks]

Question 2a

Use separation of variables to find the general solution of each of the following differential equations:

(a)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3y^4}{4x^3}$$

[4 marks]

Question 2b

(b)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2}{y(\pi - x^3)} e^{y^2}$$

[5 marks]

Question 3a

Solve each of the following differential equations for y which satisfies the given boundary condition.

(a)

$$\cos \pi x^4 \frac{\mathrm{d}y}{\mathrm{d}x} = \left(\frac{x}{y}\right)^3 \tan \pi x^4; \qquad y(0) = -3$$

[5 marks]

Question 3b

(b)

$$\left(\frac{e^{x^2}}{\cos y}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 \cos y; \qquad y(0) = \frac{3\pi}{4}$$

[6 marks]

Question 4

The evil Galactic Imperium has been spreading through the galaxy, taking over larger and larger volumes of galactic space as time goes on. The area of space controlled by the Imperium at any point in time may be modelled as a sphere centred on the capital planet Merekhty.

Representatives of the Star Rebellion are on the planet Nezal, attempting to convince the planet's inhabitants to join the rebellion. Nezal lies 16.2 kiloparsecs (kpc) away from Merekhty, however, and because of that great distance the inhabitants of the planet believe it will be a very long time before they need to worry about the Imperium's expansion.

As the Rebellion's Chief Mathematician, you have been given the job of preparing a report on the expansion of the Imperium in relation to Nezal. Based on your research, you believe that at any time, , the rate of expansion of the volume of space controlled by the Imperium, , is inversely proportional to the square of the cube root of the volume of space already controlled by the Imperium at that time.

Given that one year ago the Imperium controlled 8 cubic kiloparsecs of galactic space, whereas now it controls 2197 cubic kiloparsecs, determine how many more years it will be before Nezal falls within the Imperium's sphere of control.

[10 marks]

Question 5a

As the atoms in a sample of radioactive material undergo radioactive decay, the rate of change of the number of radioactive atoms remaining in the sample at any time t is proportional to the number, N, of radioactive atoms currently remaining. The amount of time, λ, that it takes for half the radioactive atoms in a sample of radioactive material to decay is known as the *half-life* of the material.

Let N_0 be the number of radioactive atoms originally present in a sample.

(a)
By first writing and solving an appropriate differential equation, show that the number of radioactive atoms remaining in the sample at any time $t \geq 0$ may be expressed as

$$N(t) = N_0 e^{-\frac{\ln 2}{\lambda} t}$$

[8 marks]

Question 5b

Plutonium-239, a by-product of uranium fission reactors, has a half-life of 24000 years.

(b)
For a particular sample of Plutonium-239, determine how long it will take until less than 1% of the original radioactive Plutonium-239 atoms in the sample remain.

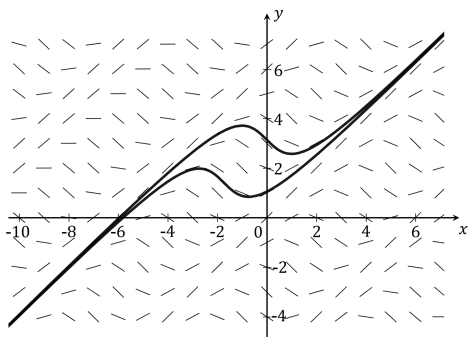
[3 marks]

Question 6a

The diagram below shows the slope field for the differential equation

$$\frac{dy}{dx} = \cos(x - y)$$

The graphs of the two solutions to the differential equation that pass through the points $\left(0, \frac{\pi}{3}\right)$ and $(0, \pi)$ are shown.



a)

Explain the relationship that must exist between x and y for $\frac{dy}{dx}$=0 to be true.

[2 marks]

Question 6b

For the two solutions given, the local minimum points lie on the straight line L_1 and the local maximum points lie on the straight line L_2.

b)

Find the equations of (i) L_1 and (ii) L_2, giving your answers in the form $y = mx + c$.

[3 marks]

Question 7a

Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{5}{\sqrt{63 + 11x^2 - 2x^4}} - \frac{2xy}{2x^2 + 7}$$

with the boundary condition $y\left(-\frac{3\sqrt{2}}{2}\right) = 1$.

(a)

Apply Euler's method with a step size of $h = 0.2$ to approximate the solution to the differential equation at $x = \frac{2 - 3\sqrt{3}}{2}$.

[3 marks]

Question 7b

It can be shown that exact solution to the differential equation with the given boundary condition is

$$y = \frac{5\sin^{-1}\left(\frac{x}{3}\right) + 4 + \frac{5\pi}{4}}{\sqrt{2x^2+7}}$$

(b)

(i)

Compare your approximation from part (a) to the exact value of the solution at $x = \frac{2-3\sqrt{2}}{2}$.

(ii)

Explain how the accuracy of the approximation in part (a) could be improved.

[3 marks]

Question 8a

A particle moves in a straight line, such that its displacement x at time t is described by the differential equation

$$x = \frac{t(t^2+1)e^{-3x}}{(2t^2+1)(2t^2+3)}, \qquad t \geq 0$$

At time $t = 0$, $x = 0$.

a)

By using Euler's method with a step length of 0.1, find an approximate value for x at time $t = 0.3$.

[3 marks]

Question 8b

b)
Solve the differential equation with the given boundary condition.

[5 marks]

Question 8c

c)
Hence find the percentage error in your approximation for x at time $t = 0.3$.

[2 marks]