5.7 Further Differential Equations

Question Paper

Course	DP IB Maths
Section	5. Calculus
Topic	5.7 Further Differential Equations
Difficulty	Very Hard

Time allowed: 150

Score: /123

Percentage: /100

Question 1a

Consider the following system of coupled differential equations

$$\dot{x} = x^2 + 2ty$$

$$\dot{y} = -6x + y + 14t$$

with the initial condition $x = -2,\ y = -12$ when $t = 0$.

a)
Use the Euler method with a step size of 0.1 to find approximations for the values of x and y when $t = 0.5$.

[6 marks]

Question 1b

b)
i)
Show that the system has a single equilibrium point at time $t = 0$ and write down its coordinates.
ii)
Find the coordinates of the equilibrium points in terms of t. Hence show that, for times $t > 0$, the system has no equilibrium points at which the values of both x and y are non-negative.

[5 marks]

Question 2a

Consider the following system of differential equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{23}{14}x + \frac{5}{7}y$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{15}{14}x + \frac{1}{7}y$$

Given that $x = 8$ and $y = 3$ when $t = 0$,

a)
use a matrix method to determine the exact solution of the system.

[10 marks]

Question 2b

b)
Hence determine the long-term ratio of the value of x to the value of y.

[2 marks]

Question 3a

The rates of change of two variables, x and y, are described by the following system of differential equations:

$$\frac{dx}{dt} = -0.3x - 2.1y \quad \frac{dy}{dt} = -8.1x + 3.3y$$

The matrix $\begin{pmatrix} -0.3 & -2.1 \\ -8.1 & 3.3 \end{pmatrix}$ has eigenvalues -3 and 6. Initially $x = 0$ and $y = 3$.

a)
Use the above information to find the exact solution to the system of differential equations.

[7 marks]

Question 3b

Use the Euler method with a step size of 0.2 to find approximations for the values of x and y when $t = 1$.

[6 marks]

Question 3c

c)

Compare the ratio of the approximations from part (b) with the ratio of x to y that you would expect in the long term based on your answer to part (a).

[2 marks]

Question 3d

d)

(i)

Find the percentage error of the approximations from part (b) compared with the exact values of x and y when $t = 1$.

(ii) Explain how the approximations found in part (b) could be improved.

[4 marks]

Question 4a

Given that the matrices $\begin{pmatrix} -0.2 & 2.4 \\ 2.4 & 1.2 \end{pmatrix}$, $\begin{pmatrix} 1.36 & 0.48 \\ 0.48 & 1.64 \end{pmatrix}$ and $\begin{pmatrix} -2.28 & 0.96 \\ 0.96 & -1.72 \end{pmatrix}$ all have $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ 3 \end{pmatrix}$ as eigenvectors,

i)
sketch the phase portrait, and
ii)
state whether the point $(0,0)$ is a stable equilibrium point or an unstable equilibrium point for each of the systems of differential equations given below.

a)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -0.2x + 2.4y \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 2.4x + 1.2y$$

[5 marks]

Question 4b

b)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 1.36x + 0.48y \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 0.48x + 1.64y$$

[5 marks]

Question 4c

c)

$$\frac{dx}{dt} = -2.28x + 0.96y \qquad \frac{dy}{dt} = 0.96x - 1.72y$$

[5 marks]

Question 5a

The behaviour of two variables, x and y, is modelled by the following system of differential equations:

$$\frac{dx}{dt} = -x - 10y \qquad \frac{dy}{dt} = 10.1x - 3y$$

where $x = 1$ and $y = 1$ when $t = 0$.

a)
Sketch the phase portrait of the system with the given initial condition.
Now consider instead the following system of differential equations

$$\frac{dx}{dt} = x - 10y \qquad \frac{dy}{dt} = 10.1x + 3y$$

with the same initial conditions.

[7 marks]

Question 5b

b)
Describe briefly how the phase portrait for this system would differ from the phase portrait drawn in part (a). Be sure to justify your answer.

[5 marks]

Question 6a

The amounts, x and y, of two reactive chemicals in a solution are modelled by the following system of differential equations:

$$\frac{dx}{dy} = 5x - 8.5y + 9.5 \qquad \frac{dy}{dt} = 4x - 5y - 5$$

a)

Find the equations of the lines on which (i) $\frac{dx}{dt}$ and (ii) $\frac{dy}{dt}$ are equal to zero, and hence determine the equilibrium point of the system.

[4 marks]

Question 6b

b)
Use the substitutions $u = x - 10$ and $v = y - 7$ to rewrite the equations as a system of coupled differential equations in u and v.

[4 marks]

Question 6c

c)
Determine the nature of the solution trajectories for the system of equations in u and v found in part (b).

[4 marks]

Question 6d

Initially $x = 1$ and $y = 1$.

d)
Use your answers to parts (a) and (c) to sketch a phase portrait showing the long-term behaviour of x and y. You may take as a given that both x and y remain non-negative for all values of t.

[4 marks]

Question 7a

Scientists are studying populations of a prey species and a predator species within a particular ecosystem. They model the two populations by the system of equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = (a - by)x \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = (cx - d)y$$

where x represents the size of the prey population, y represents the size of the predator population, and $a,\ b,\ c$ and d are all positive real parameters.

a)
Write down (i) the coordinates of the equilibrium points of the system, and (ii) the equations of the lines on which the (local) minimum and maximum values of x and y will be located. Give your answers in terms of $a,\ b,\ c$ and d as appropriate.

[4 marks]

Question 7b

Parameter is sometimes referred to as the 'prey population growth parameter', while parameter d is sometimes referred to as the 'predator population extinction parameter'.

b)
Using mathematical reasoning, explain briefly (i) why those names are suitable for parameters a and b, and (ii) what parameters b and c represent in the model.

[6 marks]

Question 7c

Let $a = 13,\ b = 5,\ c = 4$ and $d = 9$, with $x = 1$ and $y = 1$ when $t = 0$.

c)
By first using the Euler method with a step size of 0.002 to find approximations for the values of x and y between $t = 0$ and $t = 0.64$, sketch a phase portrait showing an approximate solution trajectory for the system with the given parameter values and initial conditions.

[8 marks]

Question 8a

A particle moves in a straight line, such that its displacement x metres at time t seconds is described by the differential equation

$$\ddot{x} + 5\dot{x} + 6x = 0$$

a)
Show that the second order differential equation above can be rewritten as a system of coupled first order differential equations.

[3 marks]

Question 8b

When $t = 0$, the displacement of the particle is -3 m and the velocity is $-10\ \mathrm{ms}^{-1}$.

b)
By applying Euler's method with a step size of 0.1 to the system of equations found in part (a), find approximations for (i) the displacement and (ii) the velocity of the particle at time $t = 0.5$.

[6 marks]

Question 8c

c)
By first finding the exact solution to the system of equations found in part (a), determine the percentage error of the values for the displacement and velocity at time $t = 0.5$ that were found in part (b).

[7 marks]

Question 8d

d)

Sketch the trajectory of your exact solution from part (c) on a phase diagram, showing the relationship between the particle's displacement and velocity as time *t* increases.

[4 marks]