3.9 Modelling with Vectors

Question Paper

Course	DP IB Maths
Section	3. Geometry & Trigonometry
Topic	3.9 Modelling with Vectors
Difficulty	Very Hard

Time allowed: 100

Score: /81

Percentage: /100

Question 1a

Two drones X and Y are being flown over an area of rainforest to look for signs of illegal logging. Their positions relative to the observation centre, are given by

$$\mathbf{r}_x = \begin{pmatrix} -3 \\ 1.6 \\ 2.5 \end{pmatrix} + t \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \text{ and } \mathbf{r}_y = \begin{pmatrix} 2.5 \\ 0 \\ -2 \end{pmatrix} + t \begin{pmatrix} 1.5 \\ 6 \\ 4 \end{pmatrix}$$

at time t minutes after take-off, $0 \leq t < 20$. All distances are in metres.

(a)
Verify that the two drones will not collide.

[2 marks]

Question 1b

(b)
Find the shortest distance between the two drones and the time at which it occurs.

[6 marks]

Question 1c

A third drone Z begins its flight at $t = 8$ and its position relative to the observation centre is given by $\mathbf{r}_Z = \begin{pmatrix} 2 \\ 1.5 \\ 4.5 \end{pmatrix} + t\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$

Each drone can observe a circular area of ground, A, such that $A = 1.8h^2$ where h is the height of the drone above the ground in metres.

(c)
Show that the area of ground that can be observed by drone Z five minutes after it takes off overlaps with the area of ground that can be observed by drone Y at that time.

[6 marks]

Question 2a

A car is moving at a constant speed of 15 ms^{-1} in the direction parallel to the vector $3\mathbf{i} - 6\mathbf{j}$. Two birds are perched at points $A(17, 28, 16)$ and $B(-48, 128, 26)$.

At $t = 0$, the car is located at $(2, 4, 0)$ and the bird at point A starts to fly at a constant velocity of $\frac{7\sqrt{365}}{10}$ ms^{-1}. The bird at point B begins to fly at a constant velocity in the direction of the vector $52\mathbf{i} - 60\mathbf{j} - 9k$ when $t = 1.2$.

When bird A reaches the position of $(44, -24, 4)$, both birds and the car lie in a straight line.

(a)
Find the equation of the line along which the birds and car lie.

[6 marks]

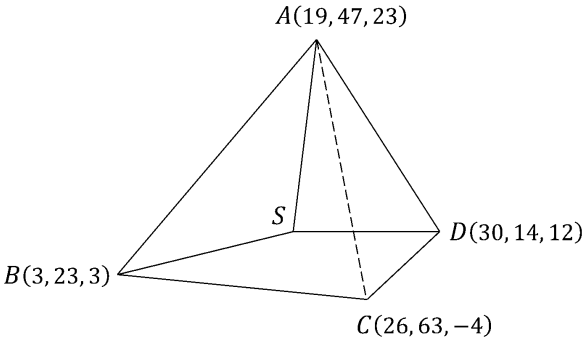
Question 2b

(b)
Find the speed at which bird B is travelling.

[6 marks]

Question 3

Consider the following diagram depicting imaginary lines connecting five points in space:



Points A, B, C and D are the locations, respectively, of the stars Arccirclus, Betacarotjuse, α-Capella and Denomineb. Point S is the location of the Stellamortis battle station, a planet-killing atrocity being built by the evil Galactic Imperium. Coordinates are given relative to an origin point in accordance with the standard x, y, z coordinate system, and the units for all coordinates are parsecs.

The forces of the Star Rebellion are prepared to launch a strike to destroy the battle station, but they are unsure of its exact location. According to data recovered from a smuggled droid, however, the following facts are known about the location of point S:

- Point S is in the First Octant of the galaxy, where x, y and z coordinates are all positive.
- The distance from point C to point S is exactly $45\sqrt{2}$ parsecs.
- Points B, C, D and S form the base of a pyramid, with its apex at point A.
- The point on *BD* closest to point *A* is also the point where the two diagonals of the pyramid's base intersect.

As the rebellion's Chief Mathematician, it is your job to use the information provided to find the exact coordinates of point S. The fate of the galaxy is in your mathematical hands!

[11 marks]

Question 4a

An oyster on the edge of a coral reef projects a microbubble into a jet stream and its subsequent motion can be modelled as a position vector. The microbubble reaches a maximum height and then moves back downwards in front of the oyster and continues down into the sea below.

The acceleration of the microbubble can be modelled by the vector

$$a = (0.4\mathrm{i} - 0.6t\mathrm{j})\,\mathrm{m\,s^{-2}}$$

Taking the origin to be the point at which the oyster is sitting, the unit vectors i and j are a displacement of 1 m along the horizontal and vertical axis of a Cartesian coordinate system respectively.

a)
Given that it takes 5 seconds until the microbubble is at the same horizontal height as the oyster again, and that the horizontal distance of the microbubble from the oyster at $t = 5$ is double that of when it is at its maximum height, find
(i)
the maximum height above the oyster that the microbubble reaches,
(ii)
the position vector of the microbubble at time, t.

[8 marks]

Question 4b

b)
The seabed is 22 metres below the level of the oyster. Find the speed of the microbubble at the moment when it hits the seabed.

[4 marks]

Question 5a

A boat is moving such that its position vector when viewed from above at time t seconds can be modelled by

$$r = \begin{pmatrix} 10 - a\sin\left(\frac{\pi t}{600}\right) \\ b\left(1 - \cos\left(\frac{\pi t}{600}\right)\right) \end{pmatrix}$$

with respect to a rectangular coordinate system from a point O, where the non-zero constants a and b can be determined. All distances are given in metres.

The boat leaves its mooring point at time $t = 0$ seconds and 5 minutes later is at the point with coordinates $(-20, 40)$.

a)
Find
(i)
the values of a and b,
(ii)
the displacement of the boat from its mooring point.

[4 marks]

Question 5b

b)
Find the velocity vector of the boat at time t seconds.

[2 marks]

Question 5c

c)
Find the time that the boat returns to its mooring point and the acceleration of the boat at this moment.

[3 marks]

Question 6a

A small stunt plane is heading in to land at an airport with acceleration given by the vector

$$a = \begin{pmatrix} -0.06 \\ -0.03t \end{pmatrix} \text{ms}^{-2}$$

The $\mathbf{i}$ component represents horizontal motion and the $\mathbf{j}$ component represents vertical motion. The start of the runway is considered the origin and the runway runs along the
horizontal axis. When $t = 10$ seconds the velocity of the plane is $0.9\mathbf{i} - 3.5\mathbf{j}\ \text{ms}^{-1}$ and the plane is 27 metres vertically above the start of the runway.

a)
Find
(i)
the time in seconds at which the stunt plane lands on the runway,
(ii)
the distance of the stunt plane from the start of the runway when it lands.

[3 marks]

Question 6b

b)
Find the speed with which the stunt plane lands.

[2 marks]

Question 6c

At the moment of landing, this particular type of stunt plane needs to have a deceleration of between 0.4 and 0.5 ms^{-2}.

c)
Decide whether the stunt plane has landed within the safe landing limits.

[1 mark]

Question 7a

Two children are observing the movement of some worms in their garden. The worms are placed on the ground at the same time and begin to move instantly. The first worm, W_1 moves with velocity at time t seconds given by the equation

$$v = \begin{pmatrix} e^{-0.4t}(a\cos t - b\sin t) \\ e^{-0.4t}(a\sin t + b\cos t) \end{pmatrix}.$$

The second worm, W_2 has position vector given by

$$r = \begin{pmatrix} 4e^{-0.5t}\cos t \\ 3e^{-0.2t}\sin t \end{pmatrix}.$$

All distances are in metres and time is in seconds.

a)
Find the velocity vector of W_2 at time t seconds.

[3 marks]

Question 7b

b)
Given that both worms are travelling parallel to each other in the same direction and at the same speed at time t=20, find
(i)
the values of a and b,
(ii)
the speed at which the two worms are travelling at this moment.

[5 marks]

Question 8a

A spider starts from the origin and begins to weave a web such that her velocity vector at time t seconds with respect to a rectangular coordinate system can be modelled by

$$v = \begin{pmatrix} a\sin(bt) + \sin(t) \\ \cos(t) - a\cos(bt) \end{pmatrix}$$

where $a < 0$ and $0 < b < 1$.

a)
Find an expression for the position vector of the spider at time t, in terms of a and b.

[4 marks]

Question 8b

b)

Given that at time $t = 9\pi$ seconds the spider is moving parallel to the y-axis with a speed of $\frac{7}{3}\text{ms}^{-1}$, find the values of a and b.

[4 marks]

Question 8c

c)

Find the earliest time at which the spider is weaving its web parallel to the x-axis.

[1 mark]