IB Maths DP

3. Geometry & Trigonometry

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3.1 Geometry Toolkit

3.1.1 Coordinate Geometry

Basic Coordinate Geometry

What are cartesian coordinates?

- **Cartesian** coordinates are basically the *x-y* coordinate system
  - They allow us to label where things are in a two-dimensional plane
- In the 2D cartesian system, the horizontal axis is labelled *x* and the vertical axis is labelled *y*

What can we do with coordinates?

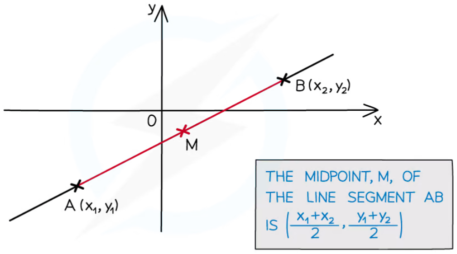
- If we have two points with coordinates (x_1, y_1) and (x_2, y_2) then we should be able to find
  - The **midpoint** of the two points
  - The **distance** between the two points
  - The **gradient** of the line between them

How do I find the midpoint of two points?

- The midpoint is the **average (middle) point**
  - It can be found by finding the middle of the *x*-coordinates and the middle of the *y*-coordinates
- The coordinates of the midpoint will be

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

  - This is given in the formula booklet under the prior learning section at the beginning

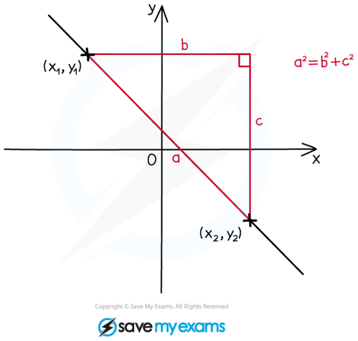


How do I find the distance between two points?

- The distance between two points with coordinates (x_1, y_1) and (x_2, y_2) can be found using the formula

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

  - This is given in the formula booklet in the *prior learning* section at the beginning
- Pythagoras' Theorem $a^2 = b^2 + c^2$ is used to find the length of a line between two coordinates
- If the coordinates are labelled A and B then the line segment between them is written with the notation [AB]



How do I find the gradient of the line between two points?

- The gradient of a line between two points with coordinates (x_1, y_1) and (x_2, y_2) can be found using the formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

  - This is given in the formula booklet under section 2.1 Gradient formula
  - This is usually known as $m = \frac{\text{rise}}{\text{run}}$

Worked Example

Point *A* has coordinates (3, -4) and point *B* has coordinates (-5, 2).

i)
Calculate the distance of the line segment *AB*.

$$A:(\underset{\overset{\nearrow}{x_1}}{3},\ \underset{\overset{\uparrow}{y_1}}{-4}) \qquad B:(\underset{\overset{\nearrow}{x_2}}{-5},\ \underset{\overset{\uparrow}{y_2}}{2})$$

Formula for distance between two points:

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Sub coordinates for A and B into the formula :

$$d = \sqrt{(3-(-5))^2 + (-4-2)^2}$$

$$= \sqrt{8^2 + (-6)^2} = \sqrt{100}$$

$$\boxed{d = 10 \text{ units}}$$

ii)
Find the gradient of the line connecting points *A* and *B*.

$$A:(\underset{\overset{\nearrow}{x_1}}{3},\ \underset{\overset{\uparrow}{y_1}}{-4}) \qquad B:(\underset{\overset{\nearrow}{x_2}}{-5},\ \underset{\overset{\uparrow}{y_2}}{2})$$

Formula for gradient of a line segment:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Sub coordinates for A and B into the formula :

$$m = \frac{2 - -4}{-5 - 3} = \frac{6}{-8} = -\frac{3}{4}$$

$$\boxed{m = -\frac{3}{4}}$$

iii)
Find the midpoint of [*AB*] .

$$A:(\underset{x_1}{3}, \underset{y_1}{-4}) \quad B:(\underset{x_2}{-5}, \underset{y_2}{2})$$

Formula for the midpoint of two coordinates:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Sub values in :

$$\text{Midpoint} = \left(\frac{3 + (-5)}{2}, \frac{-4 + 2}{2}\right) = (-1, -1)$$

$$\boxed{\text{Midpoint} = (-1, -1)}$$

Perpendicular Bisectors

What is a perpendicular bisector?

- A perpendicular bisector of a line segment cuts the line segment in half at a right angle
  - Perpendicular lines meet at right angles
  - Bisector means to cut in half
- Two lines are perpendicular if the **product of their gradients is -1**

How do I find the equation of the perpendicular bisector of a line segment?

- To find the equation of a straight line you need to find
  - The gradient of the line
  - A coordinate of a point on the line
- To find the equation of the **perpendicular bisector** of a line segment follow these steps:
  - STEP 1: Find the coordinates of the midpoint of the line segment
    - We know that the perpendicular bisector will cut the line segment in half so we can use the midpoint of the line segment as the known coordinate on the bisector
  - STEP 2: Find the gradient of the line segment
  - STEP 3: Find the gradient of the perpendicular bisector
    - This will be -1 divided by the gradient of the line segment
  - STEP 4: Substitute the gradient of the perpendicular bisector and the coordinates of the midpoint into an equation for a straight line
    - The **point-gradient** form $\boldsymbol{y - y_1 = m(x - x_1)}$ is the easiest
  - STEP 5: Rearrange into the required form
    - Either $y = mx + c$ or $ax + by + d = 0$
    - These equations for a straight line are given in the formula booklet

Worked Example

Point *A* has coordinates (4, -6) and point *B* has coordinates (8, 6). Find the equation of the perpendicular bisector to [*AB*]. Give your answer in the form $ax + by + d = 0$.

Step 1: find the coordinates of the midpoint:

$A:(\underset{x_1}{4}, \underset{y_1}{-6}) \quad B:(\underset{x_2}{8}, \underset{y_2}{6}) \qquad \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Sub values in:

$\text{Midpoint} = \left(\frac{4 + 8}{2}, \frac{-6 + 6}{2}\right) = (6, 0)$

Step 2: Find the gradient of [AB]:

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6 - -6}{8 - 4} = \frac{12}{4} = 3$

Step 3: Find the gradient of the perpendicular bisector:

$m_{\perp r} = -\frac{1}{m} = -\frac{1}{3}$

Step 4: Substitute gradient and coordinate into an equation for a straight line.

insert coordinates of the midpoint.

$(y - y_1) = m(x - x_1)$

$(y - 0) = -\frac{1}{3}(x - 6)$

Step 5: Rearrange into the form $ax + by + d = 0$

$(y - 0) = -\frac{1}{3}(x - 6) \qquad (\times -3)$

$-3y = x - 6 \qquad (+3y)$

$$\boxed{x + 3y - 6 = 0}$$

Length of an Arc

What is an arc?

- An arc is a part of the **circumference** of a circle
    - It is easiest to think of it as the crust of a single slice of pizza
- The length of an arc depends of the size of the angle at the centre of the circle
- If the angle at the centre is **less than 180°** then the arc is known as a **minor arc**
    - This could be considered as the crust of a single slice of pizza
- If the angle at the centre is **more than 180°** then the arc is known as a **major arc**
    - This could be considered as the crust of the remaining pizza after a slice has been taken away

How do I find the length of an arc?

- The length of an arc is simply a fraction of the circumference of a circle
    - The fraction can be found by dividing the angle at the centre by 360°
- The formula for the length, l, of an arc is

$$l = \frac{\theta}{360} \times 2\pi r$$

- Where θ is the angle measured in degrees
- r is the radius
- This is **in the formula booklet**, you do not need to remember it

Exam Tip

- Make sure that you read the question carefully to determine if you need to calculate the arc length of a sector, the perimeter or something else that incorporates the arc length!

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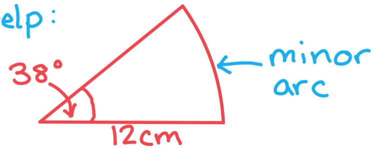
Worked Example

A circular pizza has had a slice cut from it, the angle of the slice that was cut was 38 °. The radius of the pizza is 12 cm. Find

i)
the length of the outside crust of the slice of pizza (the minor arc),

A diagram will help:



Formula for the Length of an arc:

$$l = \frac{\theta}{360} \times 2\pi r$$

Substitute:

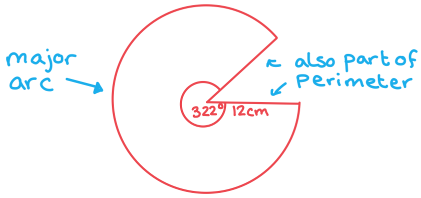
$$l = \frac{38}{360} \times 2\pi(12)$$

$$= \frac{38\pi}{15} \quad = 7.9587\ldots \text{ cm}$$

$$\boxed{\text{Length of crust } = 7.96 \text{ cm } (3\text{s.f})}$$

ii)
the perimeter of the remaining pizza.

A diagram will help:



Formula for the length of an arc:

$$l = \frac{\theta}{360} \times 2\pi r$$

Substitute:

$$l = \frac{322}{360} \times 2\pi(12)$$

$$= \frac{322}{15}\pi$$ ← length of major arc

Find perimeter:

P = major arc + radius + radius

$$= \frac{322\pi}{15} + 12 + 12 = 91.4395\ldots \text{ cm}$$

$$\boxed{\text{Perimeter} = 91.4 \text{ cm (3s.f)}}$$

Area of a Sector

What is a sector?

- A sector is a part of a circle enclosed by two radii (radiuses) and an arc
    - It is easier to think of this as the shape of a single slice of pizza
- The area of a sector depends of the size of the angle at the centre of the sector
- If the angle at the centre is **less than 180°** then the sector is known as a **minor sector**
    - This could be considered as the shape of a single slice of pizza
- If the angle at the centre is **more than 180°** then the sector is known as a **major sector**
    - This could be considered as the shape of the remaining pizza after a slice has been taken away

How do I find the area of a sector?

- The area of a sector is simply a fraction of the area of the whole circle
    - The fraction can be found by dividing the angle at the centre by 360°
- The formula for the area, A, of a sector is

$$A = \frac{\theta}{360} \times \pi r^2$$

    - Where θ is the angle measured in degrees
    - r is the radius
    - This is **in the formula booklet**, you do not need to remember it

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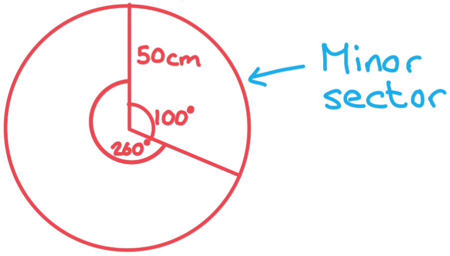
Worked Example

Jamie has divided a circle of radius 50 cm into two sectors; a minor sector of angle 100° and a major sector of angle 260°. He is going to paint the minor sector blue and the major sector yellow. Find

i)
the area Jamie will paint blue,

Start with a diagram:



Formula for the area of a sector:

$$A = \frac{\theta}{360^\circ} \times \pi r^2$$

Substitute: $A = \frac{100}{360} \times \pi \times 50^2$

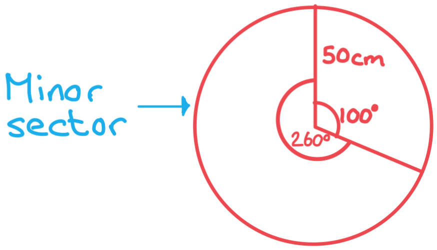
$= \frac{6250}{9} \pi$

$= 2181.66 \ldots \text{ cm}^2$

$$\boxed{\text{Blue area} = 2180 \text{ cm}^2 \text{ (3sf)}}$$

ii)
the area Jamie will paint yellow.

Start with a diagram:



Formula for the area of a sector:

$$A = \frac{\theta}{360°} \times \pi r^2$$

Substitute: $A = \frac{260}{360} \times \pi \times 50^2$

$$= \frac{16250}{9}\pi$$

$$= 5672.32\ldots \text{cm}^2$$

Yellow area = 5670 cm² (3sf)

3.2 Geometry of 3D Shapes

3.2.1 3D Coordinate Geometry

3D Coordinate Geometry

How does the 3D coordinate system work?

- In three-dimensional space we can label where any object is using the *x*-*y*-*z* coordinate system
- In the 3D cartesian system, the *x*- and *y*- axes usually represent lateral space (length and width) and the *z*-axis represents vertical height

What can we do with 3D coordinates?

- If we have two points with coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) then we should be able to find:
  - The **midpoint** of the two points
  - The **distance** between the two points
- If the coordinates are labelled A and B then the line segment between them is written with the notation [AB]

How do I find the midpoint of two points in 3D?

- The midpoint is the **average (middle) point**
  - It can be found by finding the middle of the *x*-coordinates and the middle of the *y*-coordinates
- The coordinates of the midpoint will be

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

  - This is given in the formula booklet, you do not need to remember it

How do I find the distance between two points in 3D?

- The distance between two points with coordinates $((x_1, y_1, z_1)$ and (x_2, y_2, z_2) can be found using the formula

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

  - This is given in the formula booklet, you do not need to remember it

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Worked Example

The points *A* and *B* have coordinates (-2, 1, 5) and (4, -3, 2) respectively.

i)
Calculate the distance of the line segment *AB*.

Formula for the distance of a line segment:

in formula booklet

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

$$A : (\underset{x_1}{-2}, \underset{y_1}{1}, \underset{z_1}{5}) \qquad B : (\underset{x_2}{4}, \underset{y_2}{-3}, \underset{z_2}{2})$$

Substitute:

$$d = \sqrt{(-2-4)^2 + (1-(-3))^2 + (5-2)^2}$$

$$= \sqrt{(-6)^2 + 4^2 + 3^2}$$

$$= \sqrt{36 + 16 + 9}$$

$$= \sqrt{61}$$

$$\boxed{d = 7.81 \text{ units } (3 \text{ sf})}$$

ii)
Find the midpoint of [*AB*].

Formula for the midpoint of a line segment:

in formula booklet ↙

$$MP = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

$$A : (\underset{\overset{\nearrow}{x_1}}{-2}, \underset{\overset{\uparrow}{y_1}}{1}, \underset{\overset{\uparrow}{z_1}}{5}) \qquad B : (\underset{\overset{\nearrow}{x_2}}{4}, \underset{\overset{\uparrow}{y_2}}{-3}, \underset{\overset{\uparrow}{z_2}}{2})$$

Substitute:

$$MP = \left(\frac{-2+4}{2}, \frac{1+(-3)}{2}, \frac{5+2}{2}\right)$$

$$= \left(\frac{2}{2}, -\frac{2}{2}, \frac{7}{2}\right)$$

$$\boxed{MP = (1, -1, 3.5)}$$

Volume of 3D Shapes

What is volume?

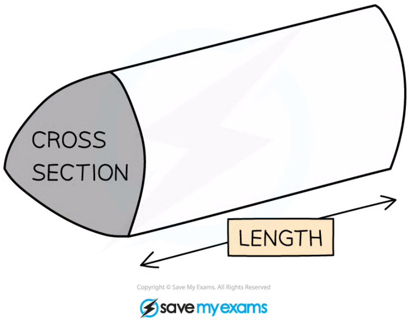
- The volume of a 3D shape is a measure of how much 3D space it takes up
  - A 3D shape is also called a **solid**
- You need to be able to calculate the volume of a number of common shapes

How do I find the volume of cuboids, prisms and cylinders?

- A prism is a 3–D shape that has two identical **base** shapes connected by parallel **edges**
  - A prism has the same base shape all the way through
  - A **prism** takes its name from its base
- To find the **volume** of any prism use the formula:

$$\text{Volume of a prism} = Ah$$

  - Where ***A*** is the area of the cross section and ***h*** is the base height
    - *h* could also be the length of the prism, depending on how it is oriented
  - This is in the formula booklet in the **prior learning** section at the beginning
  - The base could be any shape so as long as you know its area and length you can calculate the volume of any prism



- Note two special cases:
  - To find the volume of a cuboid use the formula:

$$\text{Volume of a cuboid} = \text{length} \times \text{width} \times \text{height}$$
$$V = lwh$$

  - The volume of a **cylinder** can be found in the same way as a prism using the formula:

$$\text{Volume of a cylinder} = \pi r^2 h$$

  - where r is the radius, h is the height (or length, depending on the orientation
  - Note that a cylinder is technically not a prism as its base is not a polygon, however the method for finding its volume is the same
- Both of these are **in the formula booklet** in the **prior learning** section

How do I find the volume of pyramids and cones?

- In a **right-pyramid** the apex (the joining point of the triangular faces) is vertically above the centre of the base
  - The base can be any shape but is usually a square, rectangle or triangle
- To calculate the volume of a **right-pyramid** use the formula

$$V = \frac{1}{3} Ah$$

  - Where *A* is the area of the base, *h* is the height
  - Note that the height must be **vertical to the base**

- **A right cone** is a circular-based pyramid with the vertical height joining the apex to the centre of the circular base
- To calculate the volume of a **right-cone** use the formula

$$V = \frac{1}{3} \pi r^2 h$$

  - Where r is the radius, h is the height

- These formulae are both given in the formula booklet

How do I find the volume of a sphere?

- To calculate the volume of a **sphere** use the formula

$$V = \frac{4}{3} \pi r^3$$

  - Where *r* is the radius
    - the line segment from the centre of the sphere to the surface
  - This formula is given in the formula booklet

Exam Tip

- Remember to make use of the formula booklet in the exam as all the volume formulae you need will be here
  - Formulae for basic 3D objects (cuboid, cylinder and prism) are in the **prior learning** section
  - Formulae for other 3D objects (pyramid, cone and sphere) are in the **Topic 3: Geometry** section

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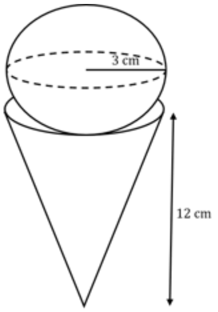
Worked Example

A dessert can be modelled as a right-cone of radius 3 cm and height 12 cm and a scoop of ice-cream in the shape of a sphere of radius 3 cm. Find the total volume of the ice-cream and cone.



Volume of a sphere: $V = \frac{4}{3}\pi r^3$ (In formula booklet)

Substitute: $r = 3$ $\Rightarrow$ $V = \frac{4}{3}\pi \times 3^3$

$= 36\pi$

Volume of a right cone: $V = \frac{1}{3}\pi r^2 h$ (In formula booklet)

Substitute: $r = 3$, $h = 12$ $\Rightarrow$ $V = \frac{1}{3}\pi(3)^2(12)$

$= 36\pi$

Total Volume $= 72\pi \text{ cm}^3$

$\boxed{\text{Total Volume} = 226\text{cm}^3 \text{ (3sf)}}$

Surface Area of 3D Shapes

What is surface area?

- The surface area of a 3D shape is the sum of the areas of all the **faces** that make up a shape
  - A **face** is one of the flat or curved surfaces that make up a 3D shape
  - It often helps to consider a 3D shape in the form of its 2D net

How do I find the surface area of cuboids, pyramids and prisms?

- Any prisms and pyramids that have polygons as their bases have only flat faces
  - The surface area is simply found by adding up the areas of these flat faces
  - Drawing a 2D net will help to see which faces the 3D shape is made up of

How do I find the surface area of cylinders, cones and spheres?

- Cones, cylinders and spheres all have curved faces so it is not always as easy to see their shape
  - The net of a **cylinder** is made up of two identical circles and a rectangle
  - The rectangle is the curved surface area and is harder to identify
  - The length of the rectangle is the same as the circumference of the circle
  - The area of the **curved surface area** is

$$A = 2\pi rh$$

    - where r is the radius, h is the height
  - This is given in the formula book in the prior learning section
  - The area of the **total surface area of a cylinder** is

$$A = 2\pi rh + 2\pi r^2$$

  - This is **not** given in the formula book, however it is easy to put together as both the area of a circle and the area of the curved surface area are given

- The net of a **cone** consists of the circular base along with the curved surface area
  - The area of the **curved surface area** is

$$A = \pi rl$$

    - Where r is the radius and l is the **slant height**
  - This is **given in the formula book**
    - Be careful not to confuse the slant height, l, with the vertical height, h
    - Note that r, h and l will create a **right-triangle** with l as the hypotenuse
  - The area of the **total surface area of a cone** is

$$A = \pi rl + \pi r^2$$

  - This is **not** given in the formula book, however it is easy to put together as both the area of a circle and the area of the curved surface area are given

- To find the surface area of a **sphere** use the formula

$$A = 4\pi r^2$$

    - where *r* is the radius (line segment from the centre to the surface)
  - This is given in the formula booklet, you do not have to remember it

Exam Tip

- Remember to make use of the formula booklet in the exam as all the area formulae you need will be here
  - Formulae for basic 2D shapes (parallelogram, triangle, trapezoid, circle, curved surface of a cylinder) are in the **prior learning** section
  - Formulae for other 2D shapes (curved surface area of a cone and surface area of a sphere) are in the **Topic 3: Geometry** section

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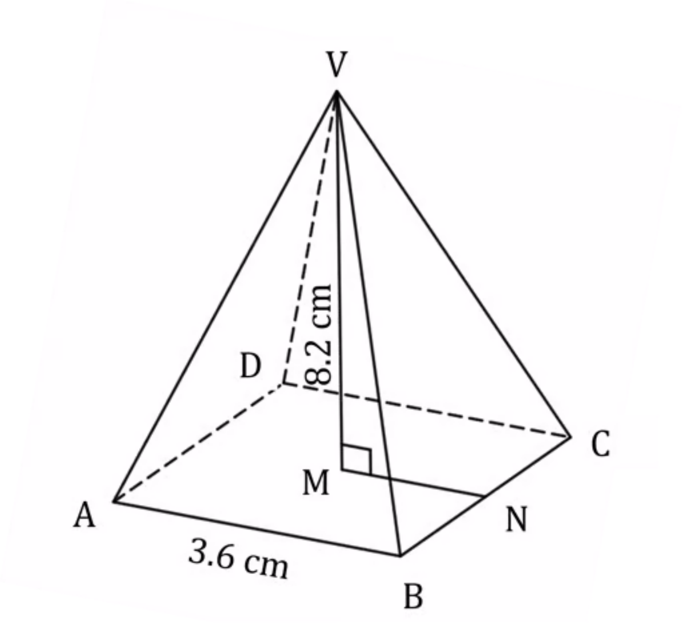
Worked Example

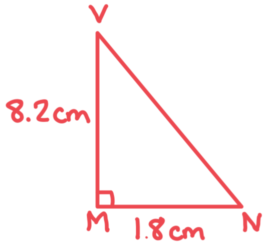
In the diagram below *ABCD* is the square base of a right pyramid with vertex *V*. The centre of the base is *M*. The sides of the square base are 3.6 cm and the vertical height is 8.2 cm.



i)

Use the Pythagorean Theorem to find the distance VN.

Sketch the triangle MNV:



↑
M is the midpoint
so MN = 3.6 ÷ 2

By the Pythagorean Theorem:

$$VN^2 = \sqrt{VM^2 + MN^2}$$

$$= \sqrt{8.2^2 + 1.8^2}$$

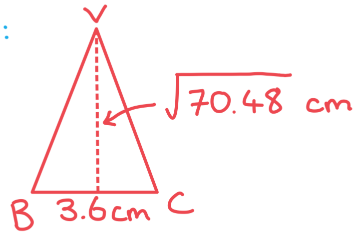
$$= \sqrt{70.48}$$

$$\boxed{VN = 8.40 \text{ cm (3sf)}}$$

ii)
Calculate the area of the triangle ABV.

Area ΔABV = area ΔBCV

Sketch ΔBCV:



Area of a triangle $= \frac{1}{2}bh$

Substitute $b = 3.6$, $h = \sqrt{70.48}$

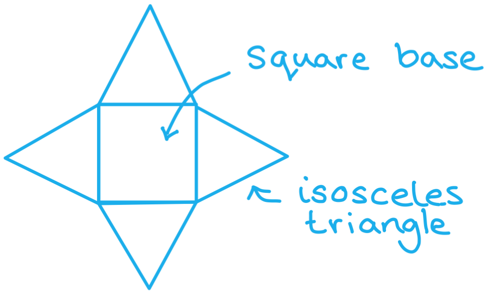
$$\text{Area} = \tfrac{1}{2}(3.6)(\sqrt{70.48})$$
$$= 15.111\ldots \text{ cm}^2$$

$$\boxed{\text{Area } \Delta ABV = 15.1 \text{ cm}^2}$$

iii)

Find the surface area of the right pyramid.

Considering the net may help:



Surface area = area square + 4(area triangle)

$$SA = 3.6^2 + 4(15.111\ldots)$$
$$= 73.405\ldots \text{ cm}^2$$

$$\boxed{SA = 73.4 \text{ cm}^2 \ (3sf)}$$

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3.3 Trigonometry

3.3.1 Pythagoras & Right-Angled Triganometry

Pythagoras

What is the Pythagorean theorem?

- Pythagoras' theorem is a formula that works for **right-angled triangles** only
- It states that for any right-angled triangle, the **square of the hypotenuse is equal to the sum of the squares of the two shorter sides**
  - The **hypotenuse** is the **longest side** in a right-angled triangle
    - It will always be **opposite** the right angle
  - If we label the hypotenuse c, and label the other two sides *a* and *b*, then Pythagoras' theorem tells us that

$$a^2 + b^2 = c^2$$

- The formula for Pythagoras' theorem is assumed prior knowledge and is **not in the formula booklet**
  - You will need to remember it

How can we use Pythagoras' theorem?

- If you know two sides of any right-angled triangle you can use Pythagoras' theorem to find the length of the third side
  - Substitute the values you have into the formula and either solve or rearrange
- To find the length of the **hypotenuse** you can use:

$$c = \sqrt{a^2 + b^2}$$

- To find the length of **one of the other sides** you can use:

$$a = \sqrt{c^2 - b^2} \text{ or } b = \sqrt{c^2 - a^2}$$

- Note that when finding the **hypotenuse** you should **add** inside the square root and when finding **one of the other sides** you should **subtract** inside the square root
- Always **check** your answer carefully to make sure that the hypotenuse is the longest side
- Note that Pythagoras' theorem questions will rarely be standalone questions and will often be 'hidden' in other geometry questions

What is the converse of the Pythagorean theorem?

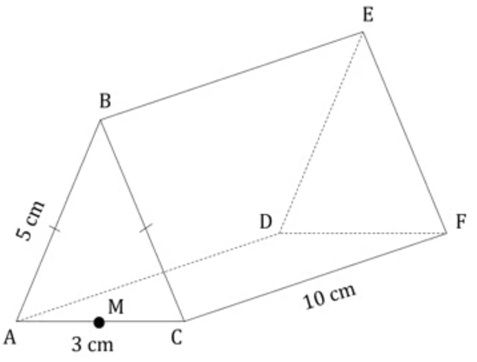
- The converse of the Pythagorean theorem states that if $a^2 + b^2 = c^2$ is true then the triangle must be a right-angled triangle
  - This is a very useful way of determining whether a triangle is right-angled
- If a diagram in a question does not clearly show that something is right-angled, you may need to use Pythagoras' theorem to check

Exam Tip

- Pythagoras' theorem pops up in lots of exam questions so bear it in mind whenever you see a right-angled triangle in an exam question!

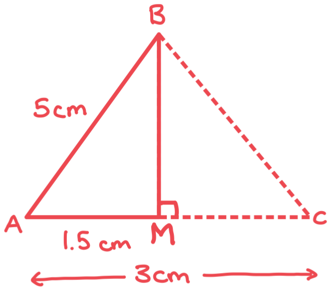
Worked Example

ABCDEF is a chocolate bar in the shape of a triangular prism. The end of the chocolate bar is an isosceles triangle where AC = 3 cm and AB = BC = 5 cm. M is the midpoint of AC. This information is shown in the diagram below.



Calculate the length BM.

Sketch the triangle ABM:



By the Pythagorean Theorem:

$$\underset{\substack{\uparrow\\\text{shorter side}}}{BM^2} = \sqrt{\underset{\substack{\uparrow\\\text{hypotenuse}}}{AB^2} - \underset{\substack{\nwarrow\\\text{shorter side}}}{AM^2}}$$

$$= \sqrt{5^2 - 1.5^2}$$

$$= \sqrt{22.75}$$

$$\boxed{BM = 4.77 \text{ cm } (3sf)}$$

Right-Angled Trigonometry

What is Trigonometry?

- Trigonometry is the mathematics of angles in triangles
- It looks at the relationship between side lengths and angles of triangles
- It comes from the Greek words *trigonon* meaning 'triangle' and *metron* meaning 'measure'

What are Sin, Cos and Tan?

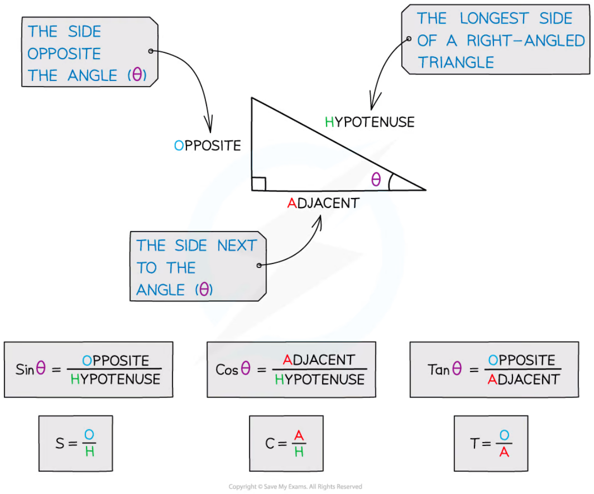
- The three trigonometric functions Sine, Cosine and Tangent come from ratios of side lengths in right-angled triangles
- To see how the ratios work you must first label the sides of a right-angled triangle in relation to a chosen angle
    - The **hypotenuse, *H*,** is the **longest side** in a right-angled triangle
        - It will always be **opposite** the right angle
    - If we label one of the other angles θ, the side opposite θ will be labelled **opposite**, ***O*,** and the side next to θ will be labelled **adjacent**, ***A***
- The functions Sine, Cosine and Tangent are the ratios of the lengths of these sides as follows

$$\text{Sin } \theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{O}{H}$$

$$\text{Cos } \theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\text{A}}{\text{H}}$$

$$\text{Tan } \theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{\text{O}}{\text{A}}$$

-
    - These are **not in the formula book**, you must remember them
- The mnemonic **SOHCAHTOA** is often used as a way of remembering which ratio is which
    - **S**in is **O**pposite over **H**ypotenuse
    - **C**os is **A**djacent over **H**ypotenuse
    - **T**an is **O**pposite over **A**djacent



How can we use SOHCAHTOA to find missing lengths?

- If you know the length of one of the sides of any right-angled triangle and one of the angles you can use SOHCAHTOA to find the length of the other sides
  - Always start by **labelling the sides** of the triangle with H, O and A
  - Choose the correct ratio by looking only at the values that you have and that you want
    - For example if you know the angle and the side opposite it (O) and you want to find the hypotenuse (H) you should use the sine ratio
  - Substitute the values into the ratio
  - Use your calculator to find the solution

How can we use SOHCAHTOA to find missing angles?

- If you know two sides of any right-angled triangle you can use SOHCAHTOA to find the size of one of the angles
- Missing angles are found using the **inverse functions**:

$$\theta = \mathrm{Sin}^{-1}\frac{O}{H}\ ,\ \theta = \mathrm{Cos}^{-1}\frac{\mathrm{A}}{\mathrm{H}}\ ,\ \theta = \mathrm{Tan}^{-1}\frac{\mathrm{O}}{\mathrm{A}}$$

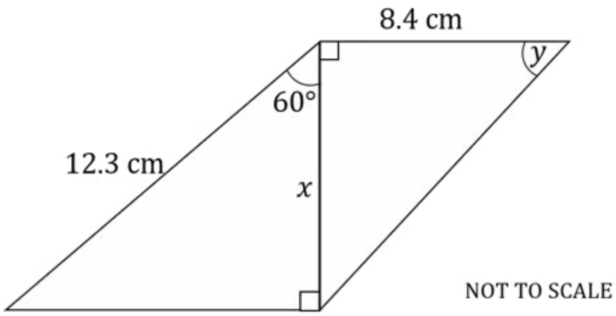
- After choosing the correct ratio and substituting the values use the inverse trigonometric functions on your calculator to find the correct answer

Exam Tip

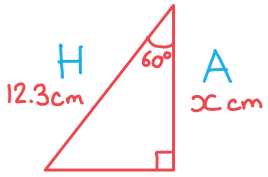
- You need to remember the sides involved in the different trig ratios as they are not given to you in the exam

Worked Example

Find the values of x and y in the following diagram. Give your answers to 3 significant figures.



Start by labelling the sides of the triangle:



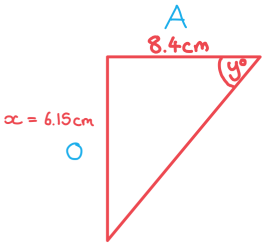
$$\text{SOH}\underbrace{\text{CAH}}\text{TOA}$$

We know H and we want to find A so we need to use $\cos\theta = \frac{A}{H}$

$$\cos 60° = \frac{x}{12.3}$$

$$x = 12.3\cos 60°$$

$$\boxed{x = 6.15\text{cm}}$$



$$\text{SOHCAH}\underbrace{\text{TOA}}$$

$$\tan y° = \frac{O}{A}$$

$$\tan y° = \frac{6.15}{8.4}$$

$$y° = \tan^{-1}\left(\frac{6.15}{8.4}\right)$$

$$\boxed{y° = 36.2° \ (3\text{ s.f.})}$$

3D Problems

How does Pythagoras work in 3D?

- 3D shapes can often be broken down into several 2D shapes
- With Pythagoras' Theorem you will be specifically looking for right-angled triangles
  - The right-angled triangles you need will have two known sides and one unknown side
  - Look for perpendicular lines to help you spot right-angled triangles
- There is a 3D version of the Pythagorean theorem formula:

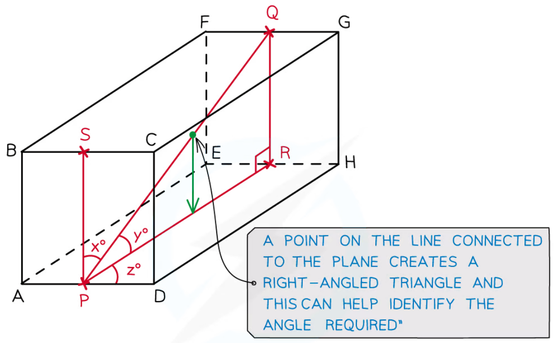
$$d^2 = x^2 + y^2 + z^2$$

  - However it is usually easier to see a problem by breaking it down into two or more 2D problems

How does SOHCAHTOA work in 3D?

- Again look for a combination of right-angled triangles that would lead to the missing angle or side
- The angle you are working with can be awkward in 3D
  - The angle between a line and a plane is not always obvious
  - If unsure put a point on the line and draw a new line to the plane
    - This should create a right-angled triangle

ANGLE BETWEEN A LINE AND A PLANE



A POINT ON THE LINE CONNECTED TO THE PLANE CREATES A RIGHT-ANGLED TRIANGLE AND THIS CAN HELP IDENTIFY THE ANGLE REQUIRED"

x° IS THE ANGLE BETWEEN THE LINE PQ AND THE PLANE ABCD (LINE PS)

y° IS THE ANGLE BETWEEN THE LINE PQ AND THE PLANE AEHD (LINE PR)

z° IS THE ANGLE BETWEEN THE LINE PR AND THE LINE AD

Exam Tip

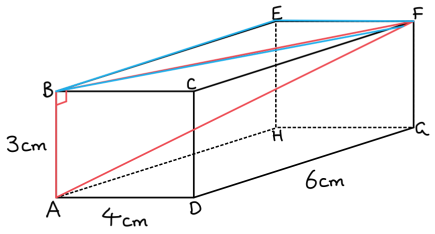
- Annotate diagrams that are given to you with values that you have calculated
- It can be useful to make additional sketches of parts of any diagrams that are given to you, especially if there are multiple lengths/angles that you are asked to find
- If you are not given a diagram, sketch a nice, big, clear one!

Worked Example

A pencil is being put into a cuboid shaped box which has dimensions 3 cm by 4 cm by 6 cm. Find, giving your answers to 1 decimal place:

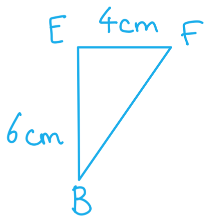
a)
the length of the longest pencil that could fit inside the box,

Draw a diagram:



The longest pencil could fit on any of the diagonals, e.g. AF.

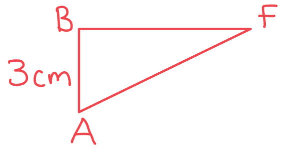
To find AF we must first find BF:



$BF^2 = 4^2 + 6^2$

$BF^2 = 16 + 36$

$BF^2 = 52$ ← can leave as BF^2 for now.



$AF^2 = 3^2 + BF^2$

$= 9 + 52$

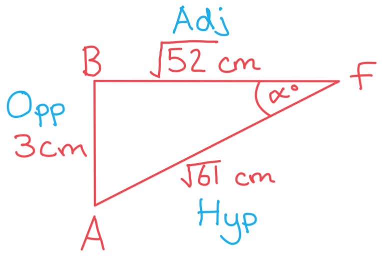
$AF^2 = 61$

$AF = \sqrt{61} = 7.8102...$

7.81 cm (3 s.f.)

b)
the angle that the pencil would make with the top of the box.

Find $A\hat{F}B$:



All three sides are known so can use any of the trig ratios.

$S^{O}{}_{H}\ C^{A}{}_{H}\ T^{O}{}_{A}$

↑

Choose $\tan\alpha = \frac{opp}{adj}$

$$\tan\alpha = \frac{3}{\sqrt{52}}$$

$$\alpha = \tan^{-1}\left(\frac{3}{\sqrt{52}}\right)$$

$$= 22.588...$$

$$\boxed{A\hat{F}B = 22.6^\circ \quad (3 \text{ s.f.})}$$

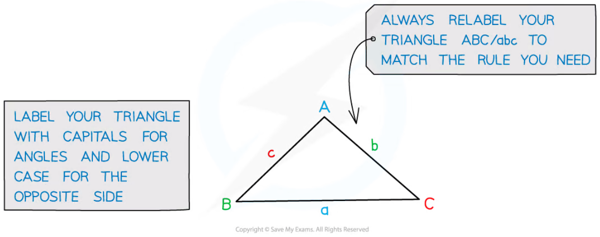
Sine Rule

What is the sine rule?

- The sine rule allows us to find missing side lengths or angles in **non-right-angled triangles**
- It states that for any triangle with angles *A*, *B* and *C*

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

  - Where
    - *a* is the side **opposite** angle *A*
    - *b* is the side **opposite** angle *B*
    - *c* is the side **opposite** angle *C*
- This formula **is in the formula booklet**, you do not need to remember it
- Sin 90° = 1 so if one of the angles is 90° this becomes SOH from **SOHCAHTOA**



How can we use the sine rule to find missing side lengths or angles?

- The sine rule can be used when you have any opposite pairs of sides and angles
- Always **start by labelling your triangle** with the angles and sides
  - Remember the sides with the lower-case letters are **opposite** the angles with the equivalent upper-case letters
- Use the formula in the formula booklet to find the **length of a side**
- To find a missing angle you can rearrange the formula and use the form

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

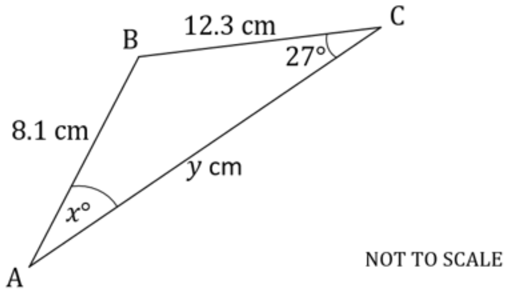
  - This is **not in the formula booklet** but can easily be found by rearranging the one given
- Substitute the values you have into the formula and solve

Exam Tip

- Remember to check that your calculator is in degrees mode!

Worked Example

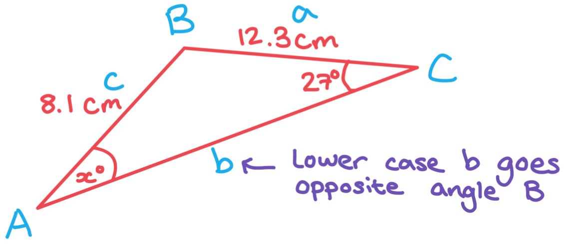
The following diagram shows triangle ABC. $\mathrm{AB} = 8.1\ \mathrm{cm}$, $\mathrm{BC} = 12.3\ \mathrm{cm}$, $\mathrm{B\widehat{C}A} = 27°$.



Use the sine rule to calculate the value of:

i)
x,

Sketch the diagram and label the sides:



Using the sine rule:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$ ← We are looking for an angle so this version is easier.

$$\frac{\sin x}{12.3} = \frac{\sin 27}{8.1}$$

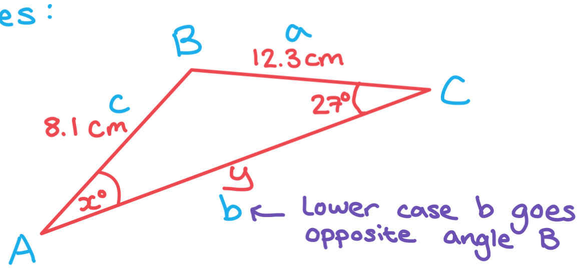
$$\sin x = \frac{12.3 \sin 27}{8.1}$$

$$x = \sin^{-1}\left(\frac{12.3 \sin 27}{8.1}\right)$$

$$\boxed{x = 43.6° \ (3s.f.)}$$

ii)
y.

Sketch the diagram and label the sides:



Find $A\hat{B}C$: $180-(27+43.582...)$

$$A\hat{B}C = 109.417...$$

Using the sine rule:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

We are looking for a side so this version is easier.

$$\frac{y}{\sin(109.417...)} = \frac{8.1}{\sin 27}$$

$$y = \frac{8.1\sin(109.417...)}{\sin 27}$$

$$\boxed{y = 16.8\text{cm} \ (3\text{ s.f.})}$$

Cosine Rule

What is the cosine rule?

- The cosine rule allows us to find missing side lengths or angles in **non-right-angled triangles**
- It states that for any triangle

$$c^2 = a^2 + b^2 - 2ab\cos C \; ; \; \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

  - Where
    - c is the side **opposite** angle C
    - *a* and *b* are the other two sides
- Both of these formulae **are in the formula booklet**, you do not need to remember them
  - The first version is used to find a missing side
  - The second version is a rearrangement of this and can be used to find a missing angle
- Cos 90° = 0 so if C = 90° this becomes **Pythagoras' Theorem**

How can we use the cosine rule to find missing side lengths or angles?

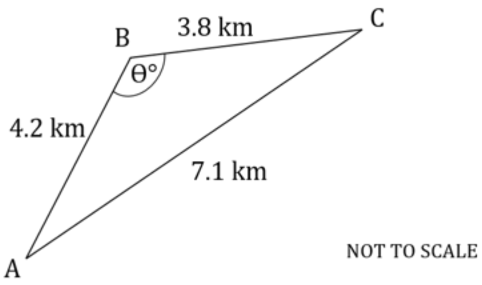
- The cosine rule can be used when you have two sides and the angle between them or all three sides
- Always **start by labelling your triangle** with the angles and sides
  - Remember the sides with the lower-case letters are **opposite** the angles with the equivalent upper-case letters
- Use the formula $c^2 = a^2 + b^2 - 2ab\cos C$ to find an unknown side
- Use the formula $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$ to find an unknown angle
  - C is the angle **between** sides *a* and *b*
- Substitute the values you have into the formula and solve

Exam Tip

- Remember to check that your calculator is in degrees mode!

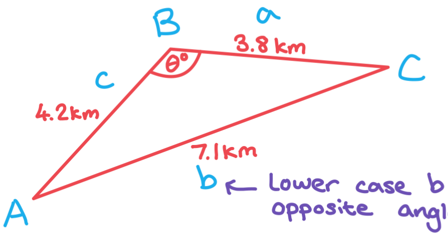
Worked Example

The following diagram shows triangle ABC. AB $= 4.2$ km, BC $= 3.8$ km, AC $= 7.1$ km.



Calculate the value of $A\hat{B}C$.

Sketch the diagram and label the sides:



← Lower case b goes opposite angle B

Using the cosine rule:

$$\cos C = \frac{a^2+b^2-c^2}{2ab}$$

← We are looking for an angle so this version is easier.

$$\cos\theta = \frac{4.2^2+3.8^2-7.1^2}{2(4.2)(3.8)}$$

$$\theta = \cos^{-1}\left(\frac{4.2^2+3.8^2-7.1^2}{2(4.2)(3.8)}\right)$$

$$= 125.04699\ldots$$

$$\boxed{\theta = 125^\circ \quad (3\text{s.f.})}$$

Area of a Triangle

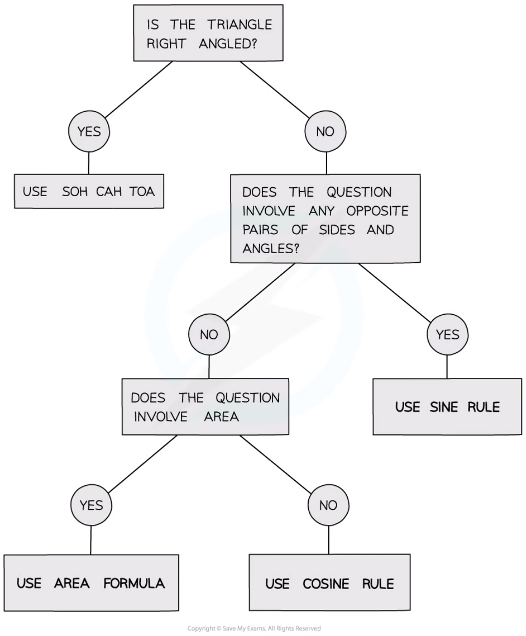
YOUR NOTES
↓

How do I find the area of a non-right triangle?

- The area of **any triangle** can be found using the formula

$$A = \frac{1}{2}ab\sin C$$

  - Where C is the angle between sides *a* and *b*
  - This formula **is in the formula booklet**, you do not need to remember it
- Be careful to label your triangle correctly so that C is always the angle **between** the two sides
- Sin 90° = 1 so if C = 90° this becomes Area = ½ × base × height

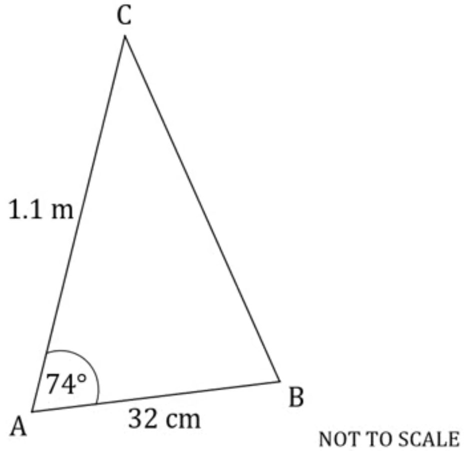


Exam Tip

- Remember to check that your calculator is in degrees mode!

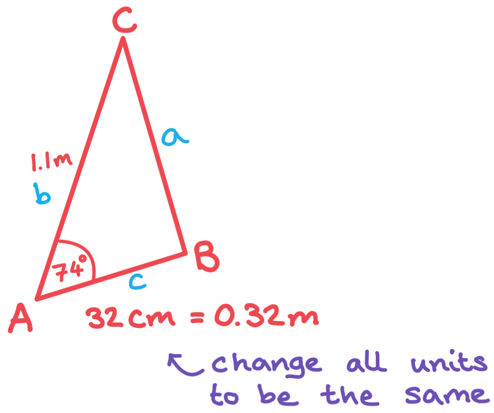
Worked Example

The following diagram shows triangle ABC. $AB = 32\text{ cm}$, $AC = 1.1\text{ m}$, $B\widehat{A}C = 74°$.



Calculate the area of triangle .

Label the sides of the triangle :



Area of a triangle : $A = \frac{1}{2}ab\sin C$

$$A = \frac{1}{2}(1.1)(0.32)\sin 74°$$

$$\boxed{A = 0.169\text{ m}^2}$$

Bearings

What are bearings?

- **Bearings** are a way of describing and using **directions** as **angles**
- They are specifically defined for use in navigation because they give a precise **location** and/or **direction**

How are bearings defined?

- There are **three rules** which must be followed every time a bearing is defined
    - They are **measured** from the **North** direction
        - An arrow showing the North line should be included on the diagram
    - They are **measured clockwise**
    - The angle is always written in **3 figures**
        - If the angle is less than 100° the first digit will be a zero

What are bearings used for?

- Bearings questions will normally involve the use of Pythagoras or trigonometry to find missing distances (lengths) and directions (angles) within navigation questions
    - You should always **draw a diagram**
- There may be a scale given or you may need to consider using a scale
    - However normally in IB you will be using triangle calculations to find the distances
- Some questions may also involve the use of angle facts to find the missing directions
- To answer a question involving **drawing bearings** the following steps may help:
    - STEP 1: Draw a diagram adding in any points and distances you have been given
    - STEP 2: Draw a North line (arrow pointing vertically up) at the point you wish to measure the bearing **from**
        - If you are given the bearing **from A to B** draw the North line at **A**
    - STEP 3: Measure the angle of the bearing given **from the North line** in the **clockwise direction**
    - STEP 4: Draw a line and add the point B at the given distance
- You will likely then need to use trigonometry to calculate the shortest distance or another given distance

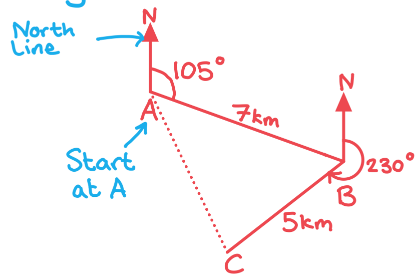
Exam Tip

- **Always** draw a big, clear diagram and annotate it, be especially careful to label the angles in the correct places!

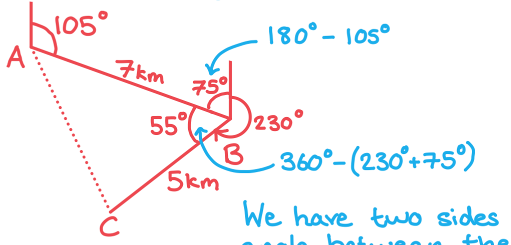
Worked Example

The point B is 7 km from A on a bearing of 105°. The distance from B to C is 5 km and the bearing from B to C is 230°. Find the distance from A to C.

Always start with a diagram:



Fill in the angles you can on the diagram



We have two sides and the angle between them so we can use the cosine rule for the third side

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$AC^2 = 7^2 + 5^2 - 2(7)(5)\cos(55°)$$

$$= 33.849\ldots$$

$$\boxed{AC = 5.82 \text{ km } (3 \text{ s.f.})}$$

Elevation & Depression

What are the angles of elevation and depression?

- If a person looks at an **object** that is not on the same horizontal line as their eye-level they will be looking at either an angle of **elevation** or **depression**
  - If a person looks **up** at an object their line of sight will be at an **angle of elevation** with the horizontal
  - If a person looks **down** at an object their line of sight will be at an **angle of depression** with the horizontal
- Angles of elevation and depression are measured **from the horizontal**
- **Right-angled trigonometry** can be used to find an angle of elevation or depression or a missing distance
- Tan is often used in real-life scenarios with angles of elevation and depression
  - For example if we know the distance we are standing from a tree and the angle of elevation of the top of the tree we can use Tan to find its height
  - Or if we are looking at a boat at to sea and we know our height above sea level and the angle of depression we can find how far away the boat is

Exam Tip

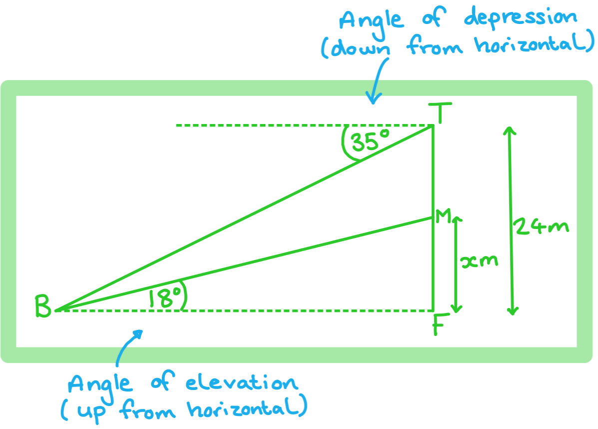
- It may be useful to draw more than one diagram if the triangles that you are interested in overlap one another

YOUR NOTES
↓

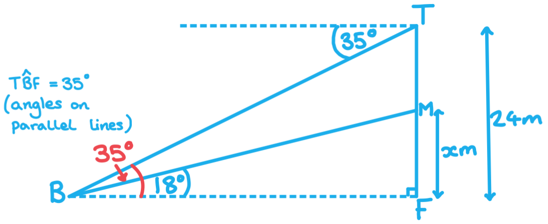
Worked Example

A cliff is perpendicular to the sea and the top of the cliff stands 24 m above the level of the sea. The angle of depression from the cliff to a boat at sea is 35°. At a point x m up the cliff is a flag marker and the angle of elevation from the boat to the flag marker is 18°.

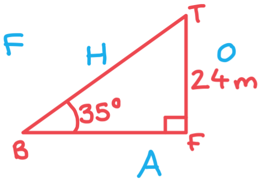
a)
Draw and label a diagram to show the top of the cliff, T, the foot of the cliff, F, the flag marker, M, and the boat, B, labelling all the angles and distances given above.



b)
Find the distance from the boat to the foot of the cliff.



Consider triangle TBF



SOHCAHTOA

we have opposite and adjacent so use Tan

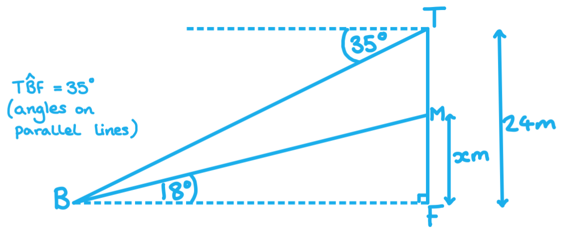
$$\text{Tan } 35° = \frac{24}{BF}$$

$$BF = \frac{24}{\text{Tan } 35°}$$

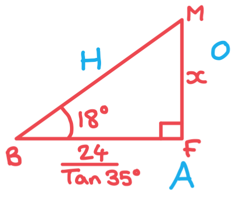
$$\boxed{BF = 34.3\text{m (3s.f.)}}$$

c)

Find the value of x.



Consider triangle FBM



SOHCAHTOA

we have opposite and adjacent so use Tan

$$\tan 18° = \frac{x}{\left(\frac{24}{\tan 35°}\right)}$$

$$x = \tan 18° \times \left(\frac{24}{\tan 35°}\right)$$

$$= 11.136...$$

$$\boxed{x = 11.1\text{ m } (3\text{s.f.})}$$

Constructing Diagrams

What diagrams will I need to construct?

- In IB you will be expected to construct diagrams based on information given
- The information will include **compass directions, bearings, angles**
    - Look out for the **plane** the diagram should be drawn in
    - It will either be **horizontal** (something occurring at sea or on the ground)
    - Or it will be **vertical** (Including height)
- Work through the statements given in the instructions systematically

What do I need to know?

- Your diagrams will be sketches, they do not need to be accurate or to scale
    - However the more accurate your diagram is the easier it is to work with
- Read the full set of instructions once before beginning to draw the diagram so you have a rough idea of where each object is
- Make sure you know your **compass directions**
    - **Due east** means on a **bearing of 090°**
        - Draw the line directly to the right
    - **Due south** means on a **bearing of 180°**
        - Draw the line vertically downwards
    - **Due west** means on a **bearing of 270°**
        - Draw the line directly to the left
    - **Due north** means on a **bearing of 360° (or 000°)**
        - Draw the line vertically upwards
- Using the above bearings for compass directions will help you to estimate angles for other bearings on your diagram

Exam Tip

- Draw your diagrams in pencil so that you can easily erase any errors

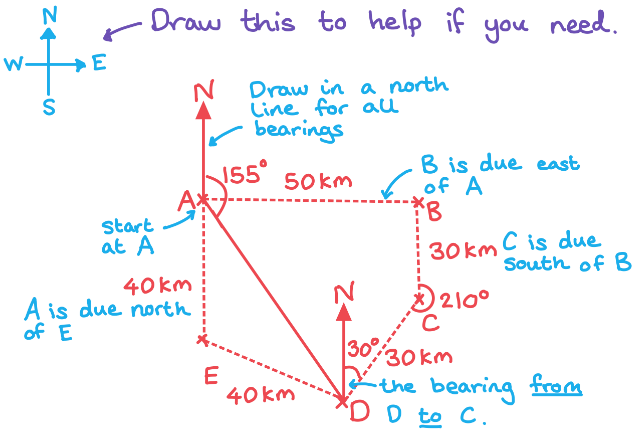
Worked Example

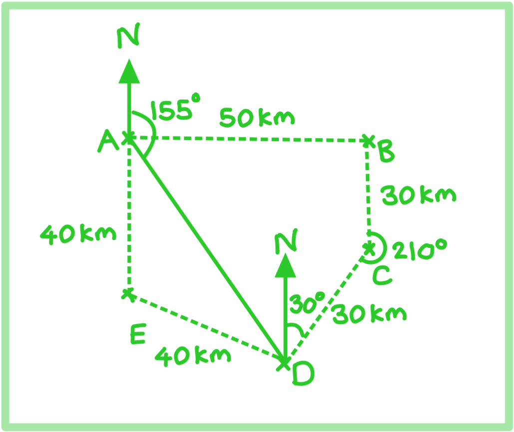
A city at B is due east of a city at A and A is due north of a city at E. A city at C is due south of B.

The bearing from A to D is 155° and the bearing from D to C is 30°.

The distance AB = 50 km, the distances BC = CD = 30 km and the distances DE = AE = 40 km.

Draw and label a diagram to show the cities A, B, C, D and E and clearly mark the bearings and distances given.





3.4 Voronoi Diagrams

3.4.1 Voronoi Diagrams

Drawing Voronoi Diagrams

What are Voronoi Diagrams?

- A **Voronoi diagram** shows the region containing the set of all points which are **closer** to one given **site** than to any other **site** on the diagram
  - A **site** is located at the coordinates of a specific place of interest on a Voronoi diagram
- It will be partitioned into a number of **regions**
  - These regions are often called **Voronoi cells** and will be **polygons**
    - There will be the same number of **regions** as **sites** on the diagram
  - For example, if a city contains five parks a Voronoi diagram could be drawn for that city dividing it into five regions based on their closest park
- The **edges** of each region will be the **perpendicular bisector** of two of the sites
  - The **edges** may also be called **boundaries**
- The **vertices** of each region are the **intersections** of **three** of these perpendicular bisectors
  - The perpendicular bisectors of three individual points will always intersect at the point that is **equidistant** from the three points

How are Voronoi diagrams drawn for three sites?

- You will **not** be expected to draw a Voronoi diagram from scratch, however you should understand how one is constructed
  - First, the perpendicular bisector of the line segment joining each pair of sites will be constructed
    - These should be constructed using dashed lines as only a part of each line will be needed for the final diagram
  - The **points of intersection** of these perpendicular bisectors will create the **vertices**
  - Each perpendicular bisector should stop when it meets another perpendicular bisector
    - Remove the part of the perpendicular bisector that is not in the region of the two sites
    - No perpendicular bisector should cross over another
    - This will form the **regions**, or **cells**

How are Voronoi diagrams drawn for more than three sites?

- It is challenging to draw a Voronoi diagram from scratch if it has **more than three sites**
- In this case it is easiest to draw the Voronoi diagram for three sites first and then add the next sites one by one following these steps

  - STEP 1: The fourth site will be in one of the cells containing an existing site

    - Draw the perpendicular bisector of the line segment between these two sites

  - STEP 2: Stop this new line at the point where it meets an existing **boundary** in the Voronoi diagram
  - STEP 3: There will now be an existing edge in the region of the new site

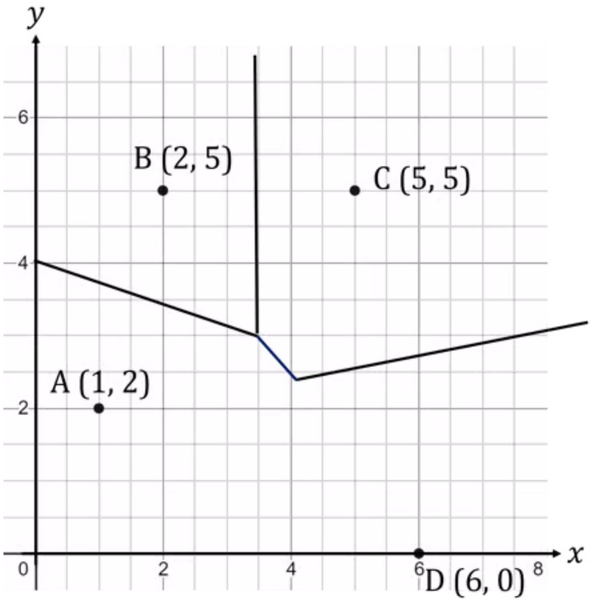
        - This should be **shortened** to meet the new boundary
    - STEP 4: The fourth site will now be in the same cell as a different existing site
        - Draw the perpendicular bisector of the line segment between these two sites
        - This is the step you will most likely carry out in an exam
- You may be asked to find the **equation of a missing edge**
    - This will mean finding the equation of the **perpendicular bisector** between the **two sites** that are both within **one region**
- You may be asked to add the **location of a missing site** to the Voronoi diagram
    - This will mean using the given **edge** of one or two of the regions and finding the second site that would make this edge a perpendicular bisector
        - Draw a perpendicular line from the site to the edge
        - Check the distance of this line and then continue it on the other site of the edge for the same distance
        - This will be the location of your new site
    - You may need to find the gradients of the edges you have and then use the negative reciprocal to find the gradient of the perpendicular bisector of the current and new site

Exam Tip

- Make sure that you have a straight edge and an eraser with you in the exam so that any perpendicular bisectors that you draw are clear and any mistakes that are made can be erased
- If you are asked to adjust a given Voronoi diagram and a perpendicular bisector that needs to be removed or shortened, you can put a series of little lines along it to indicate that it is crossed out

Worked Example

The Voronoi diagram below shows sites A, B, C and D.



a)

Explain how you know that the Voronoi diagram is incomplete.

The Voronoi diagram has four sites but only three Voronoi cells.

b)

Find the equation of the line which would complete the Voronoi cell containing site A.

Give your answer in the form $ax + by + d = 0$ where $a, b, d \in \mathbb{Z}$.

Sites A and D are both in the same region so find the perpendicular bisector of the line segment connecting A and D.

$A:(1,2) \quad D:(6,0)$

Find the midpoint:

$$MP = \left(\frac{1+6}{2}, \frac{2+0}{2}\right) = (3.5, 1)$$

gradient AD

Perpendicular gradient.

$$m_{AD} = \frac{0-2}{6-1} = -\frac{2}{5} \quad \therefore \quad m_{\perp AD} = \frac{5}{2}$$

Sub MP and $m_{\perp AD}$ into equation for a straight line:

$$y - y_1 = m(x - x_1)$$

$$y - 1 = \frac{5}{2}\left(x - \frac{7}{2}\right)$$

multiply by 2 to remove the fraction and rearrange

$$2y - 2 = 5x - \frac{35}{2}$$

$$4y - 4 = 10x - 35$$

$$\boxed{10x - 4y - 31 = 0}$$

Interpreting Voronoi Diagrams

What is a Voronoi diagram used for?

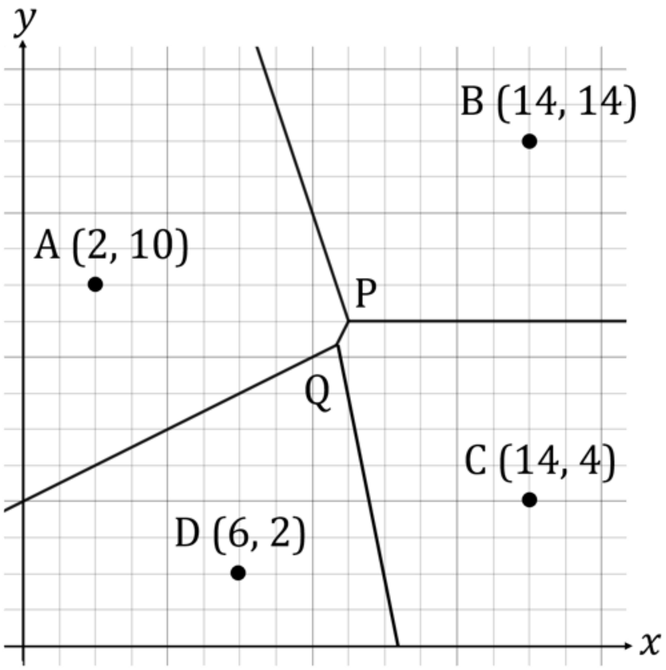
- Voronoi diagrams are often used in land management to work out where the best location would be according to where sites are already situated
- They can show where to put something to make sure that it is
    - Closest to a particular site
    - Closer to one site than another
    - Equidistant from two or three specific sites
    - As far as possible from any other site

What do I need to know about Voronoi diagrams?

- You may be asked to find the shortest distance from a point to its closest site
    - Use Pythagoras' Theorem to find the distance between the given coordinate and the site in the same region as it
    - If the coordinate is on an edge then there will be two sites **equidistant** from it
- You may be asked to find the point which is furthest from any of the sites
    - This will be one of the vertices
    - To choose which vertex look at which is the centre of the **largest empty circle**
- You may be asked to estimate the success of a new site
    - This is done by looking at the data for the **nearest site**
    - The prediction for the new site would be assumed to be the same
        - This is called **nearest neighbour interpolation**

Worked Example

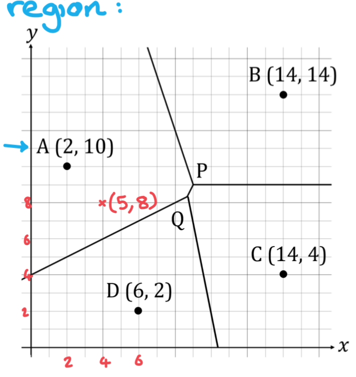
The Voronoi diagram below shows the four sites A, B, C and D with coordinates (2, 10), (14, 14), (14, 4), and (6, 2) respectively. 1 unit represents 10 km.



i)
State which site a new business opening at the coordinate (5, 8) should look at to predict future sales.

Plot the point and look for the site in the same region:

The new business is in the same region as site A.



Site A

ii)
Find the shortest distance from the point (5, 8) to its nearest site.

The point (5, 8) is closest to site A.

Formula for distance between two points:

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$P:(\underset{x_1}{5}, \underset{y_1}{8}) \quad A:(\underset{x_2}{2}, \underset{y_2}{10})$$

Sub coordinates:

$$d = \sqrt{(2-5)^2 + (10-8)^2}$$

$$= \sqrt{(-3)^2 + (2)^2} = \sqrt{13}$$

$$\text{distance} = \sqrt{13} \times 10\text{km} = 36.055\ldots \text{ km}$$

$$\boxed{\text{distance} = 36.1\text{km} \ (3\text{s.f.})}$$

3.4.2 Toxic Waste Dump Problem

Toxic Waste Dump Problem

What is the toxic waste dump problem?

- The **toxic waste dump problem** is the name given to the general idea of finding the point on a **Voronoi diagram** which is furthest from any of the **sites**
  - A **site** is the coordinates of a specific place of interest on a Voronoi diagram
- It is given this name because of the common problem of finding a place to put a toxic waste dump that is **equally far** away from any inhabited area
  - For example, if a province contains five towns a Voronoi diagram could be used to find the point within the province which is furthest from each town
- The toxic waste dump problem is more of an idea than a specific problem
  - The same concept could be applied to other contexts such as
    - Finding a position for a new supermarket that is equally far from all competitors
    - Finding a place to plant a new tree that is equally far from other trees competing for water resources
    - Finding the quietest place to enjoy a picnic that is equally far from other noisy groups of people
  - Note that the term **equally far** is used in all of the above examples

How is a Voronoi diagram used to find the furthest point from any site?

- Within any Voronoi diagram the furthest point from any site will always be either
  - one of the cell vertices, or
  - somewhere on a boundary of the diagram
- In an IB exam, the solution will always be one of the **cell vertices**
- To find the furthest point you will need to consider each of the cell vertices separately and find which one is furthest from all of the sites
- This is done by constructing the **largest empty circle**

What is the largest empty circle?

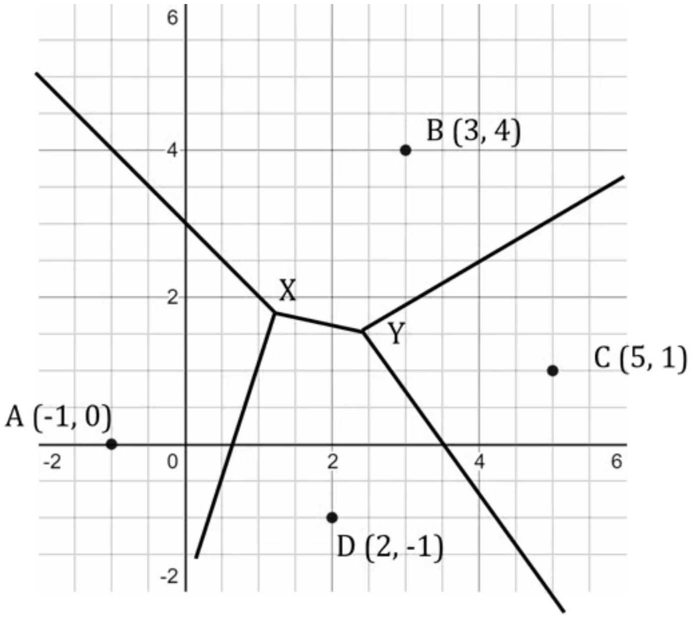
- The **largest empty circle** is the largest possible circle constructed on a Voronoi diagram that contains **no sites**
- The **centre** of the circle will be one of the vertices of a **cell** or **region**
  - The **vertices** of each region are the **intersections** of **the boundaries**
- The **radius** of the circle will be the **distance** from the vertex to the closest site
  - The closest site will be on the circumference
  - Use Pythagoras' Theorem to find the distance
- There may be a **scale** to convert the distance found on the Voronoi diagram into a distance in real life
  - For example if the scale is 1 unit represents 5 km then 5 units represents 25 km

Exam Tip

- The solution to the toxic waste dump will always be one of the points of intersection between the perpendicular bisectors, so you need to know the coordinates of these points
  - Remember that you can use your GDC to solve a pair of the simultaneous equations quickly if you know the equations of two of the perpendicular bisectors that intersect at that point

Worked Example

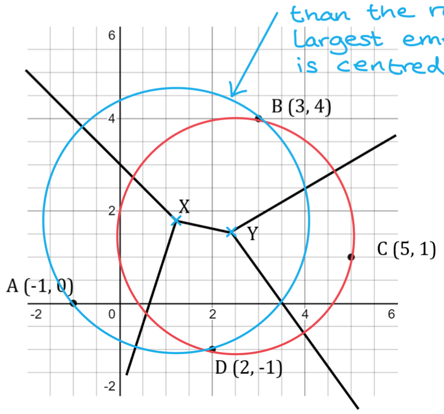
The Voronoi diagram below shows four cities at the sites A, B, C and D. The coordinates of the points X and Y are $\left(\frac{5}{4}, \frac{7}{4}\right)$ and $\left(\frac{5}{2}, \frac{3}{2}\right)$ respectively.



Determine the optimal position where a toxic waste site could be located and, given that 1 unit represents 50 km, find the distance from this point to its nearest city.

The optimal position would be at the point X or Y

Draw the largest possible circle centred at X and Y.



The blue circle is larger than the red so the largest empty circle is centred at X

Choose any of the sites on the circle to find the distance

Formula for distance between two points:

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \qquad X:(\underset{x_1}{\tfrac{5}{4}}, \underset{y_1}{\tfrac{7}{4}}) \qquad B:(\underset{x_2}{3}, \underset{y_2}{4})$$

Sub coordinates

$$d = \sqrt{\left(\tfrac{5}{4} - 3\right)^2 + \left(\tfrac{7}{4} - 4\right)^2} = \sqrt{\left(\tfrac{7}{4}\right)^2 + \left(\tfrac{9}{4}\right)^2} = \sqrt{\tfrac{65}{8}}$$

$= 2.8504\ldots$ units

distance $= 2.8504\ldots \times 50\text{km} = 142.52\ldots$ km

$$\boxed{\text{distance} = 143\text{ km } (3\text{ s.f.})}$$