1.6 Further Complex Numbers

Question Paper

Course	DP IB Maths
Section	1. Number & Algebra
Topic	1.6 Further Complex Numbers
Difficulty	Hard

Time allowed: 100

Score: /80

Percentage: /100

Question 1a

Consider the equation $z^2 + pz - 2p - 1 = 0$, where $z \in \mathbb{C}$, $p \in \mathbb{R}$.

(a)

Find the value of p for which one of the two distinct roots is $z_1 = 2 + \sqrt{3}\mathrm{i}$.

[4 marks]

Question 1b

(b)

Find the range of values of p for which the equation has two distinct, real roots.

[4 marks]

Question 2a

Let $w = 2 - iz$, where $w, z \in \mathbb{C}$.

a)
Find w when
i)
$z = 2e^{\frac{\pi}{2}i}$
ii)
$z = \sqrt{2}e^{\frac{\pi}{4}i}$

[4 marks]

Question 2b

b)
On an Argand diagram the point z can be transformed to the point w by two transformations. Describe the two transformations and the order in which they are applied.

[4 marks]

Question 2c

c)
Hence, or otherwise, find the value of z when $w = 1 + \mathrm{i}$.

[2 marks]

Question 3

Consider $z = \text{cis}\ \theta$ where $z \in \mathbb{C}$, $z \neq 1$.

Show that $\text{Re}\left(\frac{1+z}{1-z}\right) = 0$.

[5 marks]

Question 4a

It is given that $z_1 = 2\ \text{cis}\left(\frac{\pi}{4}\right)$ and $z_2 = \sqrt{2}\ \text{cis}\left(\frac{n\pi}{12}\right)$, $n \in \mathbb{Z}^+$.

a)

Giving your answers in the form $re^{i\theta}$, $r \geq 0$, $-\pi < \theta \leq \pi$, use technology to find the values of

i)

z_1^3

ii)

$(z_1 z_2)^3$, for $n = 2$.

[5 marks]

Question 4b

b)

Find the least value of n such that $z_1 z_2 \in \mathbb{R}^+$.

[3 marks]

Question 5a

Let $z = 1 + \mathrm{i}$.

a)

Express z in the form $z = ae^{\mathrm{i}b}$, where $a, b \in \mathbb{R}$, giving the exact values of a and b.

[2 marks]

Question 5b

b)

Let $w_1 = e^{ix}$ and $w_2 = zw_1$.

i)

Write $w_1 + w_2$ in the form $e^{\mathrm{i}x}(c + id)$.

ii)

Hence, find $Re(w_1 + w_2)$ in the form $A\cos(x + a)$, giving the exact value of A, where $A > 0$ and $0 < \alpha < \frac{\pi}{2}$.

[6 marks]

Question 6a

Consider the complex numbers $w_1 = \frac{z_1}{z_2}$, $z_1 = \frac{\sqrt{2}e^{-\frac{\pi}{3}\mathrm{i}}}{3}$ and $z_2 = 2 - 2\sqrt{3}\mathrm{i}$.

(a)
Express
(i)
z_1 in the form $a + b\mathrm{i}$
(ii)
z_2 in the form $r\,\mathrm{cis}\,\theta$, where $r > 0$ and $-\pi < \theta < \pi$.

[3 marks]

Question 6b

(b)
Find the exact value of w_1.

[2 marks]

Question 6c

(c)
Find $w_2 = z_1 z_2$, giving your answer in the form $r\,\mathrm{cis}\,\theta$, where $r > 0$ and $-\pi < \theta < \pi$.

[2 marks]

Question 6d

(d)
Without drawing an Argand diagram, describe the geometrical relationship between z_1 and z_2.

[1 mark]

Question 7a

$$z = \frac{\sqrt{3}}{2}\mathrm{i} - \frac{1}{2}$$

a)
Use technology to find all the powers z^n.

[5 marks]

Question 7b

(b)
Find the area of the shape made by the powers z^n when plotted on an Argand diagram.
Give your answer as an exact value.

[3 marks]

Question 8a

Let $z = \cos\theta + \mathrm{i}\sin\theta$.

a)
Write down the value of zz^*.

[2 marks]

Question 8b

Let $z_1 = r_1 e^{\mathrm{i}\theta}$ and $z_2 = r_2^{\,\mathrm{i}\left(\theta + \frac{\pi}{2}\right)}$

b)
Prove the results
i)
$\mathrm{Re}(z_1 + z_2) = r_1\cos\theta - r_2\sin\theta$
ii)
$\mathrm{Im}(z_1 + z_2) = r_1\sin\theta + r_2\cos\theta$

[3 marks]

Question 8c

c)
Using technology, or otherwise, show that

$$\mathrm{Re}(2e^{\mathrm{i}5x} + 6e^{\mathrm{i}(5x+1)} = 7.28\cos(0.77 + 5x)$$

[4 marks]

Question 9a

The current, I, in an AC circuit can be modelled by the equation $I = a\cos(bt - c)$ where b is the frequency and c is the phase shift.

Two AC voltage sources of the same frequency generate currents $I_A = 12\cos(bt)$ and $I_B = 15\cos\left(bt - \frac{\pi}{4}\right)$.

a)
Write down the maximum value and phase shift of the two currents I_A and I_B when they are each connected to the circuit alone.

[2 marks]

Question 9b

The two AC voltage sources are connected to the circuit at the same time and the total current can be expressed as $I_A + I_B$.

b)
Write down the maximum value and phase shift of $I_A + I_B$.

[5 marks]

Question 10a

The height of a wave in metres, relative to a particular boat, can be modelled by the function $h(t) = 0.5\sin(2t)$, where t is the time in seconds. Observers on the boat are tracking a jumping dolphin. The height of the dolphin's jumps can be modelled by the function $j(t) = 2\sin(2t - 0.5)$.

a)
Find an expression for the height the dolphin can reach, at time t seconds, when the height of the dolphin's jump is affected by the height of the waves. Give your answer in the form $f(t) = A\sin(bt - c)$

[4 marks]

Question 10b

b)
Use technology to find the time when the dolphin first reaches its maximum height and write down the maximum height the dolphin reaches.

[3 marks]

Question 10c

c)
Find the time interval in the first two seconds when the height of dolphin will be above the wave

[2 marks]