5.10 Differential Equations

Question Paper

Course	DP IB Maths
Section	5. Calculus
Topic	5.10 Differential Equations
Difficulty	Very Hard

Time allowed: 130

Score: /108

Percentage: /100

Question 1

Consider the first-order differential equation

$$\frac{dy}{dx} + \frac{1}{2x} = \sin 3x \cos 3x$$

Solve the equation given that $y = 0$ when $x = \frac{\pi}{2}$, giving your answer in the form $y = f(x)$.   .

[5 marks]

Question 2a

Use separation of variables to solve each of the following differential equations

(a)

$$\frac{dy}{dx} = \frac{3y^4}{4x^3}$$

[4 marks]

Question 2b

(b)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2}{y(\pi - x^3)} e^{y^2}$$

[5 marks]

Question 3a

Solve each of the following differential equations for y which satisfies the given boundary condition, giving your answers in the form $y = f(x)$.

(a)

$$\cos \pi x^4 \frac{\mathrm{d}y}{\mathrm{d}x} = \tan \pi x^4 \left(\frac{x}{y}\right)^3; \qquad y(0) = -3$$

[5 marks]

Question 3b

(b)

$$e^{x^2}\text{cosec}\, y\frac{\mathrm{d}y}{\mathrm{d}x}=x\sin y; \quad y(0)=\frac{3\pi}{4}$$

[6 marks]

Question 4a

As the atoms in a sample of radioactive material undergo radioactive decay, the rate of change of the number of radioactive atoms remaining in the sample at any time t is proportional to the number, N, of radioactive atoms currently remaining. The amount of time, λ, that it takes for half the radioactive atoms in a sample of radioactive material to decay is known as the *half-life* of the material.

Let N_0 be the number of radioactive atoms originally present in a sample.

(a)
By first writing and solving an appropriate differential equation, show that the number of radioactive atoms remaining in the sample at any time $t \geq 0$ may be expressed as

$$N(t)=N_0 e^{-\frac{\ln 2}{\lambda}t}$$

[8 marks]

Question 4b

Plutonium-239, a by-product of uranium fission reactors, has a half-life of 24000 years.

(b)
For a particular sample of Plutonium-239, determine how long it will take until less than 1% of the original radioactive Plutonium-239 atoms in the sample remain.

[3 marks]

Question 5a

Consider the standard logistic equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP(a - P)$$

where P is the size of a population at time $t \geq 0$, and where k and a are positive constants. Let the population at time $t = 0$ be denoted by P_0.

(a)
Write down the solution to the logistic equation in the case where $P_0 = a$, using mathematical reasoning to justify your answer.

[2 marks]

Question 5b

(b)
In the case where $P_0 \neq \alpha$, show that the solution to the logistic equation is

$$P(t) = \frac{aAe^{akt}}{1 + Ae^{akt}}$$

where A is an arbitrary constant.

[8 marks]

Question 5c

(c)
In the case where $P_0 \neq \alpha$, write down an expression for A in terms of a and P_0.

.

[2 marks]

Question 5d

(d)
In the case where $P_0 \neq 0$, determine the behaviour of P as t becomes large.

[3 marks]

Question 5e

(e)
In the case where $0 < 2P_0 < a$, determine the value of t at which the initial population will have doubled. Your answer should be given explicitly in terms of a, k and P_0.

[4 marks]

Question 6

Solve the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} - y = \frac{xy^2}{y^2\sin\left(\frac{y}{x}\right) - x^2\cos\left(\frac{x}{y}\right)}$$

[8 marks]

Question 7a

Consider the differential equation

$$x^2 y' = y^2 + 3xy - 8x^2$$

with the boundary condition $y(1) = -3$.

(a)
Solve the differential equation for y which satisfies the given boundary condition, giving your answer in the form $y = f(x)$.

[9 marks]

Question 7b

(b)

Determine the asymptotic behaviour of the graph of the solution as x becomes large.

[3 marks]

Question 8

Solve the differential equation

$$(4x^2+1)y' + y = \frac{1 - x + 4x^2 - 4x^3}{\sqrt{e^{\arctan 2x}}}$$

[7 marks]

Question 9a

Consider the differential equation

$$\frac{dy}{dx} = \frac{5}{\sqrt{63 + 11x^2 - 2x^4}} - \frac{2xy}{2x^2 + 7}$$

with the boundary condition $y\left(-\frac{3\sqrt{2}}{2}\right) = 1$.

(a)

Apply Euler's method with a step size of $h = 0.2$ to approximate the solution to the differential equation at $x = \frac{2 - 3\sqrt{3}}{2}$.

[3 marks]

Question 9b

(b)

Solve the differential equation analytically, for y which satisfies the given boundary condition.

[7 marks]

Question 9c

(c)

(i)

Compare your approximation from part (a) to the exact value of the solution at $x = \frac{2-3\sqrt{2}}{2}$.

(ii)

Explain how the accuracy of the approximation in part (a) could be improved.

[3 marks]

Question 10a

A particle moves in a straight line, such that its displacement x at time t is described by the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\sin t}{1+\cos^2 t}, \qquad 0 \le t \le 3$$

At time $t = 1.6,\ x = 1$.

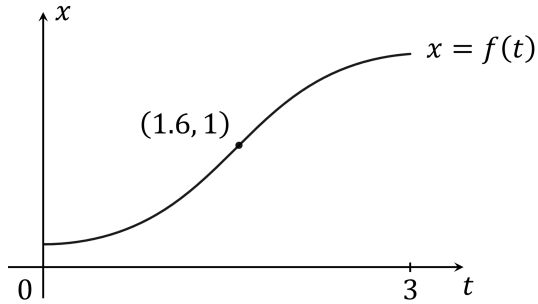
(a)

By using Euler's method with a step length of 0.04, find an approximate value for x at time $t = 1.8$.

[3 marks]

Question 10b

The diagram below shows a graph of the exact solution $x = f(t)$ to the differential equation with the given boundary condition.



Given that the graph of $x = f(t)$ has exactly one point of inflection, find the exact value of the t-coordinate of the point of inflection.

[7 marks]

Question 10c

(c)

Hence determine whether the approximation found in part (a) will be an overestimate or an underestimate for the true value of x when $t = 1.8$. Be sure to use mathematical reasoning to justify your answer.

[3 marks]