5.2 Further Differentiation

Question Paper

Course	DP IB Maths
Section	5. Calculus
Topic	5.2 Further Differentiation
Difficulty	Very Hard

Time allowed: 130

Score: /104

Percentage: /100

Question 1a

Find an expression for the derivative of each of the following functions:

a)
$f(x) = (12x^2 - 7)e^{-2x}$

[2 marks]

Question 1b

b)
$g(x) = \dfrac{\tan 3x}{4 - 5x^3}$

[3 marks]

Question 1c

c)
$h(x) = (\ln(2x^2 - x - 2))^5$

[3 marks]

Question 1d

d)

$$j(x) = \frac{2x^{-\frac{3}{4}}}{1 - x^{\frac{3}{5}}}$$

[4 marks]

Question 2a

Find an expression for the derivative of each of the following functions:

a)
$f(x) = (3x - 1)e^{\sin x}D$

[3 marks]

Question 2b

b)
$g(x) = \ln(\cos(x^2 - 1))$

[3 marks]

Question 2c

c)

$$h(x) = \frac{-\sin(e^{-x})}{e^{x\cos x}}$$

[4 marks]

Question 2d

d)

$$j(x) = \tan\left(\frac{1}{x^2\sqrt[3]{x}}\right)$$

[4 marks]

Question 3

Consider the function f defined by $f(x) = -x + \frac{2}{3}\sin^3 x,\ x \in \mathbb{R}$.

Show that f is decreasing everywhere on its domain.

[1 mark]

Question 4a

Consider the function g defined by $g(x) = e^{2x} - 2x,\ x \in \mathbb{R}$.

Point A is the point on the graph of g for which the x-coordinate is $\ln\sqrt{3}$.

a)
Show that the equation of the tangent to the graph of g at point A may be expressed in the form

$$y = 4x - 3(\ln 3 - 1)$$

[4 marks]

Question 4b

Point B is the point on the graph of g at which the normal to the graph is vertical.

b)
Show that the coordinates of the point of intersection between the tangent to the graph of g at point A and the tangent to the graph of g at point B are

$$\left(\frac{3\ln 3 - 2}{4}, 1\right)$$

[5 marks]

Question 5

Consider the function h defined by $h(x) = \sin 3x + e^{3\sqrt{3}x}\cos 3x,\ x \in \mathbb{R}$.

Show that the normal line to the graph of h at $x = \frac{\pi}{9}$ intercepts the y-axis at the point

$$\left(0, \frac{2\pi}{27} + \frac{\sqrt{3} + e^{\frac{\pi\sqrt{3}}{3}}}{2}\right)$$

[9 marks]

Question 6

Let $f(x) = g(x)h(x)$, where g and h are real-valued functions such that

$$g(x) = \ln\left(\frac{x}{3}\right)h(x)$$

for all $x > 0$.

Given that $h(3) = a$ and $h'(3) = b$, where $a \neq 0$, find the distance between the y-intercept of the tangent to the graph of f at $x = 3$ and the y-intercept of the normal to the graph of f at $x = 3$. Give your answer in terms of a and/or b as appropriate.

[8 marks]

Question 7a

Consider the curve with equation $y = \cos(kx)e^{\sin(kx)}$ defined for all $x \in \mathbb{R}$, where $k \neq 0$ is a positive integer.

a)

For the case where $k = 1$, find the number of points in the interval $-\frac{\pi}{2} \leq x < \frac{3\pi}{2}$ at which the curve has a horizontal tangent.

[1 mark]

Question 7b

b)

i)

Show algebraically that in general the x-coordinates of the points at which the curve has horizontal tangents will be the solutions to the equation

$$\sin^2(kx) + \sin(kx) - 1 = 0$$

ii)

Hence, for the case where $k = 1$, find the x-coordinates of the points identified in part (a).

[7 marks]

Question 7c

c)

i)

By considering $\frac{d^2y}{dx^2}$, show algebraically that in general the x-coordinates of the points at which the curve is neither concave up nor concave down will be the solutions to the equation

$$\sin(kx)\cos(kx) = 0$$

ii)

Hence, for the case where $k = 1$, find the -coordinates of the points of inflection on the curve in the interval $-\frac{\pi}{2} \leq x < \frac{3\pi}{2}$.

[8 marks]

Question 7d

d)
In terms of k, state in general how many (i) turning points and (ii) points of inflection the curve will have in the interval $-\frac{\pi}{2} \leq x < \frac{3\pi}{2}$. Give a reason for your answers.

[2 marks]

Question 8

Let $f(x) = \frac{g(x)}{h(x)}$, where g and h are well-defined functions with $h(x) \neq 0$ anywhere on their common domain.

By first writing $f(x) = g(x)[h(x)]^{-1}$, use the product and chain rules to show that

$$f(x) = \frac{h(x)g'(x) - g(x)h'(x)}{[h(x)]^2}$$

[4 marks]

Question 9a

Consider the function f defined by $f(x) = e^{x^k}$, $x \in \mathbb{R}$, where $k \geq 1$ is a positive integer.

a)
Show that the graph of f will have no points of inflection in the case where $k = 1$.

[2 marks]

Question 9b

b)
Show that, for $k \geq 2$, the second derivative of is given by

$$f''(x) = kx^{k-2}(kx^k + k - 1)e^{x^k}$$

[5 marks]

Question 9c

c)
Hence show that the graph of f will only have points of inflection in the case where k is an odd integer greater than or equal to 3. In that case, give the exact coordinates of the points of inflection, giving your answer in terms of k where appropriate.
In your work you may use without proof the fact that for odd integers k with $k \geq 3$

$$-1 < \sqrt[k]{-\frac{k-1}{k}} < -\frac{1}{2}$$

[7 marks]

Question 10

A small conical flask, in the shape of a right cone stood on its flat base, is being filled with perfume via a small hole at its vertex. The cone has a height of 6 cm and a radius of 2 cm.

Perfume is being poured into the flask at a constant rate of 0.3 cm^3s^{-1}.

Find the rate of change of the depth of the perfume in the flask at the instant when the flask is half full by volume.

[7 marks]

Question 11

A large block of ice is being prepared for use by a team of ice sculptors. The block is in the shape of a cuboid with the ratio of its length to width to height being equal to 1 : 2 : 5. The block melts uniformly such that its surface area decreases at a constant rate, losing $k\ m^2$ of surface area every hour. You may assume that as the block melts, its shape remains a cuboid with the dimensions in the same ratio to each other as in the original cuboid.

The block of ice is considered stable enough to be sculpted so long as the loss of volume due to melting does not exceed a rate $0.05\ \mathrm{m}^3$ per hour.

Find, in terms of k, the volume of the largest block of ice that can be used for ice sculpting under such conditions.

[8 marks]