5.1 Differentiation

Question Paper

Course	DP IB Maths
Section	5. Calculus
Topic	5.1 Differentiation
Difficulty	Hard

Time allowed: 100

Score: /82

Percentage: /100

Question 1a

The equation of a curve is $y = x - \frac{9}{x} + 8$ for $x > 0$.

(a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

[2 marks]

Question 1b

The gradient of the tangent to the curve at point A is 2.

(b) Find the coordinates of point A.

[3 marks]

Question 1c

(c) Find the equation of the normal to the curve at point A. Give your answer in the form $ax + by + d = 0$.

[3 marks]

Question 2a

The volume of a sphere of radius r is given by the formula $V = \frac{4}{3}\pi r^3$.

(a) Find $\frac{\mathrm{d}V}{\mathrm{d}r}$.

[1 mark]

Question 2b

(b) Find the rate of change of the volume with respect to the radius when $r = 5$.
Give your answer in terms of π.

[2 marks]

Question 2c

(c) Show that $\frac{\mathrm{d}V}{\mathrm{d}r}$ is an increasing function for all relevant values of r.

[3 marks]

Question 3a

A curve has the equation

$$f(x) = \frac{1}{3}x^3 - 2x^2 - 4x + \frac{31}{3}$$

Points A and B are the two points on the curve where the gradient is equal to 1, and the x-coordinate of A is less than zero.

(a) Find the coordinates of points A and B.

[3 marks]

Question 3b

(b) Find the equations of

(i) the tangent to the curve at point A

(ii) the normal to the curve at point B.

[5 marks]

Question 3c

Point C is the point of intersection of the two lines found in part (b).

(c) Find the coordinates of point C.

[2 marks]

Question 4

The gradient of the tangent to the curve with equation $f(x) = ax^2 + 2x + 9$ at the point $(-2, b)$ is 14.

Find the values of a and b.

[5 marks]

Question 5a

Patroclus, a would-be Olympic javelin thrower, throws a javelin during a training session. The height of the javelin's point can be modelled by the equation

$$h(t) = 1.75 + 20.2t - 4.90t^2$$

where t is the time, in seconds, that has passed since the javelin was released, and $h(t)$ is the height of the javelin above the ground, in metres.

(a) Find $h'(t)$.

[2 marks]

Question 5b

(b) (i) Find the stationary point for $h(t)$.

(ii) Justify that the stationary point is a maximum point.

[6 marks]

Question 5c

(c) Find the greatest vertical distance that the javelin's point travels above the height from which it was released.

[1 mark]

Question 6a

Check, Mate! is a company that produces luxury chess sets for discerning chess set connoisseurs. The company's profits $P(x)$, in thousands of UK pounds (£1000), can be modelled by the function

$$P(x) = 0.32x^3 - 12.4x^2 + 150x - 480$$

where x is the number of chess sets (in hundreds) sold per year. Because of manufacturing constraints, the maximum number of chess sets that the company can sell in a year is 2500.

(a) (i) State why there is no need to consider values of x greater than 25.

(ii) Sketch a graph of $P(x)$ for $0 \le x \le 25$.

[3 marks]

Question 6b

(b) (i) Find the stationary points on the graph, and the numbers of chess sets sold and profits that correspond to those points.

(ii) Find the maximum profit that the company can make in a year, and the number of chess sets the company must sell to make that profit.

[5 marks]

Question 6c

(c) Calculate

(i) the average rate of change of $P(x)$ between $x = 5$ and $x = 6$

(ii) the instantaneous rate of change of $P(x)$ at $x = 5$.

In each case include the units, and explain the meaning of the value you find.

[5 marks]

Question 6d

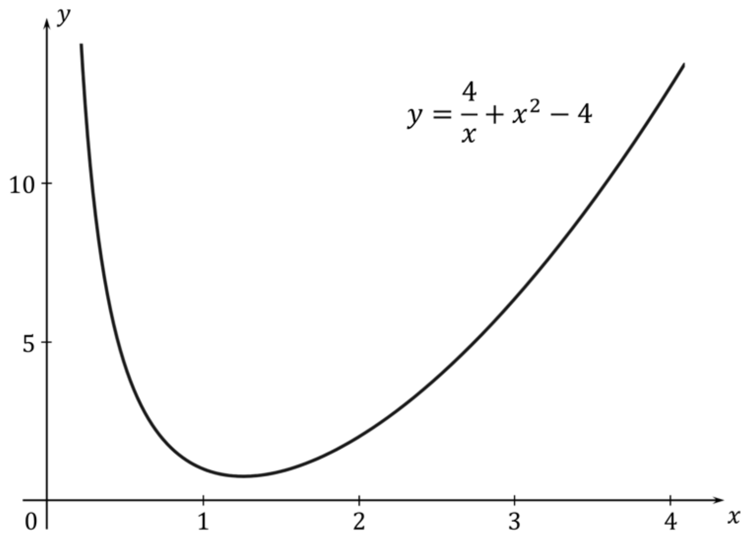
(d) State the values of x for which the instantaneous rate of change of $P(x)$ is negative.
Explain the meaning of this result.

[3 marks]

Question 7a

The diagram below shows a part of the graph of the function $y = f(x)$, where

$$f(x) = \frac{4}{x} + x^2 - 4, \qquad x > 0$$



(a) Calculate the instantaneous rate of change of $f(x)$ when $x = 2$.

[2 marks]

Question 7b

(b) Calculate the average rate of change of $f(x)$ between $x = 2$ and

(i) $x = 3$

(ii) $x = 2.5$

(iii) $x = 2.25$

[4 marks]

Question 7c

(c) Explain what would happen if you continued to calculate the average rates of change in part (b), moving the second x value closer and closer to 2 each time.

[2 marks]

Question 8a

A manufacturing company is producing tins that must have a capacity of 470 cm^3. The tins are in the shape of a cylinder with a height of h cm and a base radius of r cm.

(a) Show that the surface area of the cylinder in cm^2, including the two circular ends, may be written as

$$A = 2\pi r^2 + \frac{940}{r}$$

[4 marks]

Question 8b

(b) Sketch the graph of $A = 2\pi r^2 + \dfrac{940}{r}$.

[2 marks]

Question 8c

The company would like to minimise the amount of metal used to make the tins.

(c) (i) Find the stationary point on the graph of $A = 2\pi r^2 + \dfrac{940}{r}$, and justify that it is a minimum point.

(ii) Hence find the minimum possible surface area for the tin, and the base radius that corresponds to that minimum area.

[5 marks]

Question 8d

A commercially available tin of chopped tomatoes on sale in the UK has a capacity of $470\ \text{cm}^3$ and a base radius of 3.7 cm.

(d) Determine the percentage difference between the surface area of that tin of chopped tomatoes and the minimum possible surface area for a tin with the same capacity.

[3 marks]

Question 9

Two numbers, x and y, are such that $x > y$ and the difference between the two numbers is 7.

Find the minimum possible value of the product xy, and the values of x and y that correspond to that minimum value.

[6 marks]