# 5.1 Differentiation

## Question Paper

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| Course | DP IB Maths |
| Section | 5. Calculus |
| Topic | 5.1 Differentiation |
| Difficulty | Very Hard |

**Time allowed:** 110

**Score:** /88

**Percentage:** /100

## Question 1a

A curve is given by the equation

$$y = \frac{1}{6}x^3 - \frac{3}{8}x^2 - \frac{3}{2}x + 4$$

(a) Determine the coordinates of the points on the curve where the gradient is 2. You must show all your working, and give your answers as exact fractions.

**[6 marks]**

## Question 1b

(b) Find the range of values for $x$ for which the curve is increasing.

**[3 marks]**

### Question 2a

An engineer is designing a right cone that is to be produced on a 3D printer. The cone has a base radius of $r$ cm and a height of $h$ cm, and while the radius may vary freely the height must always be 7 cm more than the radius.

(a) Write down, in terms of $r$ only, the formula for the volume of the cone.

**[2 marks]**

### Question 2b

(b) Find the exact value of the radius at the point where the instantaneous rate of change of the volume with respect to the radius is $\frac{5\pi}{3}$ cm$^3$/cm.

**[5 marks]**

## Question 3a

A curve has the equation

$$f(x) = 2x^3 + \frac{3}{x} - 4$$

Points A and B are the two points on the curve where the gradient is equal to 3, and the $x$-coordinate of A is less than zero.

(a) Find the coordinates of points A and B.

**[3 marks]**

## Question 3b

(b) Find the equations of

(i) the tangent to the curve at point A

(ii) the normal to the curve at point B.

**[5 marks]**

## Question 3c

Point C is the point of intersection of the two lines found in part (b).

(c) Find the coordinates of point C. Give your answers as exact fractions.

**[3 marks]**

## Question 4

A curve has equation $f(x) = ax^2 + bx + c$.

The gradient of the tangent to the curve at the point $(-3, d)$ is 25.

The gradient of the tangent to the curve at the point $(2, -1)$ is $-5$.

Find the values of $a$, $b$, $c$ and $d$.

**[7 marks]**

## Question 5a

A newly-commissioned attack submarine is performing a series of manoeuvres to test its propulsion and steering systems. The vertical position of the submarine relative to sea level (where sea level is represented by $h = 0$) is given by the equation

$$h(t) = 0.0125t^3 - 1.03t^2 + 16.6t - 165, \qquad 0 \leq t \leq 60$$

where $t$ is the time, in minutes, that has passed since the submarine began its manoeuvres, and $h(t)$ is the vertical position of the submarine in metres.

(a) Find the stationary points for $h(t)$.

**[5 marks]**

### Question 5b

(b) For each of the stationary points found in part (a), determine whether the point is a maximum point or a minimum point. Justify your answer in each case.

**[4 marks]**

### Question 5c

(c) Explain why, in order to find the maximum and minimum depths reached by the submarine in the interval $0 \leq t \leq 60$, it is not sufficient merely to consider the stationary points found in part (a).

**[1 mark]**

### Question 5d

(d) Find the greatest vertical distances that the submarine travels in the interval $0 \leq t \leq 60$ above and below the depth from which it started its manoeuvres.

**[2 marks]**

## Question 6a

Muggins! is a company that produces luxury cribbage boards for discerning collectors of pub game paraphernalia. For sales of between 0 and 100 cribbage boards in a month, the company's profits $P(x)$, in thousands of UK pounds (£1000), can be modelled by the function

$$P(x) = 4.53x^2 - 8.51$$

where $x$ is the number of cribbage boards (in hundreds) sold during the month. For sales of between 100 and 1000 cribbage boards in a month, the corresponding formula is

$$P(x) = 0.02x^3 - \frac{9}{x} + 5$$

Because of manufacturing constraints, the maximum number of cribbage boards that the company can sell in a month is 1000.

(a) (i) Confirm that both formulae give the same profit for sales of 100 cribbage boards in a month.

(ii) State the ranges of $x$ values for which each formula is valid.

**[2 marks]**

## Question 6b

(b) On the same set of axes, sketch the two profit functions. Each function should only be sketched over the interval of $x$ values for which it is valid.

**[3 marks]**

### Question 6c

(c) Show that the combined profit function sketched in part (b) is an increasing function for all valid $x$ values greater than zero.

**[4 marks]**

### Question 6d

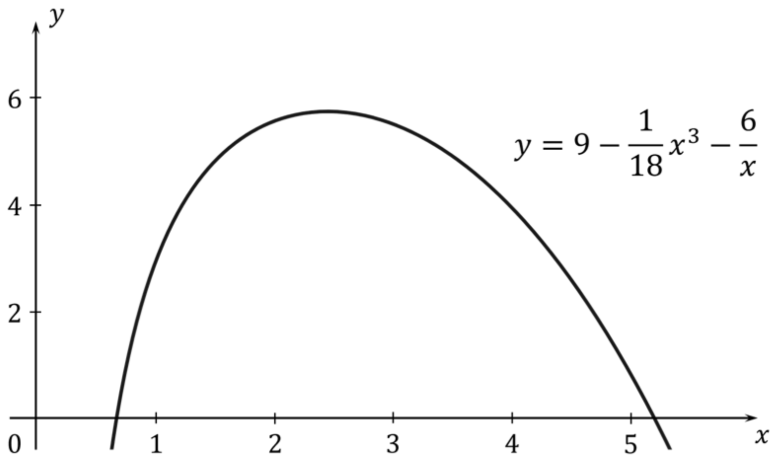
(d) Considering only values of $x$ for which $P(x) > 0$, find the value of $x$ for which the instantaneous rate of change of $P(x)$ is a minimum. Give the value of the corresponding instantaneous rate of change, and explain the meaning of that value in context.

**[5 marks]**

## Question 7a

The diagram below shows a part of the graph of the function $y = f(x)$, where

$$f(x) = 9 - \frac{1}{18}x^3 - \frac{6}{x}, \qquad x > 0$$




(a) Calculate the average rate of change of $f(x)$ between $x = 3$ and

(i) $x = 4$

(ii) $x = 3.5$

(iii) $x = 3.25$

**[4 marks]**

## Question 7b

(b) Explain what would happen to the values of the average rates of change in part (b) if you continued to calculate them, moving the second $x$ value closer and closer to 3 each time.

**[3 marks]**

## Question 8a

An artist is producing large pieces of sculpture for an art installation. Each piece is in the form of a cylinder with base radius $r$ metres, on top of which is a hemisphere with the same radius as the cylinder's base radius. The hemisphere is fitted exactly to the top of the cylinder, so that the circular bottom of the hemisphere lines up exactly with the circular top of the cylinder.

Every side of each piece of sculpture must be painted, so the artist is eager to find a design for his sculptures such that, for any given volume of a piece of sculpture, the total surface area will be the minimum possible.

(a) Show that for a piece of sculpture with volume $k\pi$ m$^3$, the minimum surface area occurs when

$$r = \sqrt[3]{\frac{3k}{5}}$$

**[9 marks]**

## Question 8b

(b) Find the minimum possible surface area for a piece of sculpture with volume $\frac{40}{3}\pi \text{ m}^3$.

Give your answer as an exact value.

**[2 marks]**

### Question 9a

Two numbers, $x$ and $y$, are such that $x > y$ and the difference between the two numbers is $k$, where $k$ is a positive constant.

(a) Find the minimum possible value of the sum $x^2 + 3y^2$, and the values of $x$ and $y$ that correspond to that minimum value. Your answers should be given in terms of $k$.

**[6 marks]**

### Question 9b

(b) Justify that your answer in part (a) is a minimum value.

**[4 marks]**