5.11 MacLaurin Series

Question Paper

Course	DP IB Maths
Section	5. Calculus
Topic	5.11 MacLaurin Series
Difficulty	Hard

Time allowed: 120

Score: /94

Percentage: /100

Question 1a

Consider the general Maclaurin series formula

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + ... + \frac{x^n}{n!}f^{(n)}(0) + ...$$

(where $f^{(n)}$ indicates the n^{th} derivative of f).

(a)
Explain why the formula cannot be used to calculate a Maclaurin expansion for $\ln x$.

[2 marks]

Question 1b

(b)
Use the formula to find the first five non-zero terms of the Maclaurin series for $\ln(1+x)$.

[4 marks]

Question 1c

(c)
Hence approximate the value of $\ln 2$.
(i)
by substituting the value $x = 1$
(ii)
by substituting the value $x = -\frac{1}{2}$.

[4 marks]

Question 1d

(d)

(i)

Compare the approximations found in part (c) to the exact value of $\ln 2$.

(ii)

Explain briefly the reason for the difference in accuracy between the two approximations.

[4 marks]

Question 1e

(e)

Use the general Maclaurin series formula to show that the general term of the Maclaurin series for $\ln(1+x)$ is

$$\frac{(-1)^{n+1}x^n}{n}, \qquad n \geq 1$$

[3 marks]

Question 2a

(a)
Find the first four non-zero terms of the Maclaurin series for $\cos 4x$ in ascending powers of x.

[3 marks]

Question 2b

(b)
Hence approximate the value of $\cos 3$ and compare this approximation to the exact value.

[3 marks]

Question 2c

(c)
Explain how the accuracy of the Maclaurin series approximation in part (b) could be improved.

[1 mark]

Question 3a

(a)

Find the Maclaurin series for $e^{2x}\ln(1+x)$ in ascending powers of x, up to and including the term in x^4.

[5 marks]

Question 3b

(b)

Hence find the first four terms of the Maclaurin series for $e^{2x}\left(2\ln(1+x)+\frac{1}{1+x}\right)$ in ascending powers of x.

[3 marks]

Question 4a

(a)

Use the general Maclaurin series formula to find the first four terms of the Maclaurin series for $\frac{1}{2+3x}$ in ascending powers of x.

[4 marks]

Question 4b

(b)

Confirm that the answer to part (a) matches the first four terms of the binomial theorem expansion of $\frac{1}{2+3x}$.

[3 marks]

Question 4c

(c)

Find the Maclaurin series for $\ln(2+3x)$ in ascending powers of x, up to and including the term in x^4.

[3 marks]

Question 4d

(d)

Find the derivative of $\ln(2+3x)$, and confirm that the series found in parts (a) and (c) reflect the relationship between $\ln(2+3x)$ and $\frac{1}{2+3x}$ that is thereby implied.

[3 marks]

Question 5a

(a)

(i)

Write down the first five non-zero terms of the Maclaurin series for $\arctan x$ in ascending powers of x.

(ii)

Hence find an approximation for the value of the integral

$$\int_0^1 \arctan x \, \mathrm{d}x$$

[3 marks]

Question 5b

(b)

Use integration by parts to show that

$$\int \arctan x \, \mathrm{d}x = x \arctan x - \frac{1}{2}\ln(1+x^2) + c$$

[4 marks]

Question 5c

(c)

Hence determine the exact value of $\int_0^1 \arctan x \, \mathrm{d}x$, and compare it to the approximation found in part (a)(ii).

[3 marks]

Question 6a

(a)

Use the binomial theorem to find a Maclaurin series for the function f defined by

$$f(x) = \sqrt{4 - 9x^2}$$

Give the series in ascending powers of x up to and including the term in x^6.

[4 marks]

Question 6b

(b)
State any limitations on the validity of the series expansion found in part (a).

[2 marks]

Question 6c

(c)
Use the answer to part (a) to estimate the value of $\sqrt{3.91}$, and compare the accuracy of that estimated value to the actual value of $\sqrt{3.91}$.

[4 marks]

Question 7a

Consider the differential equation

$$y' = x^2 - 3y^2$$

together with the initial condition $y(0) = 1$.

(a)
Find expressions for y'', y''', $y^{(4)}$ and $y^{(5)}$. Each should be given in terms of x and y and of lower-order derivatives of y.

[4 marks]

Question 7b

Let $f(x)$ be the solution to the differential equation above with the given boundary condition, so that $y = f(x)$.

(b)
Find the first six terms in ascending powers of x of the Maclaurin series for $f(x)$.

[7 marks]

Question 7c

(c)
Hence approximate the value of y when $x = 0.1$.

[2 marks]

Question 8a

Consider the differential equation

$$y' = -2xy$$

with the initial condition $y(0) = 2$.

a)
By first finding expressions for y'', y''', $y^{(4)}$, $y^{(5)}$ and $y^{(6)}$ in terms of x, y and lower-order derivatives of y, find a Maclaurin series for the solution to the differential equation with the given boundary condition, in ascending powers of x up to and including the term in x^6.

[9 marks]

Question 8b

(b)
Solve the differential equation with the given boundary condition analytically to find an exact solution in the form $y = f(x)$.

[4 marks]

Question 8c

(c)
Find the first four non-zero terms of the Maclaurin series for the answer to part (b), and confirm that they match those in the answer to part (a).

[3 marks]