5.7 Further Differential Equations

Question Paper

Course	DP IB Maths
Section	5. Calculus
Topic	5.7 Further Differential Equations
Difficulty	Medium

Time allowed: 100

Score: /83

Percentage: /100

Question 1a

Use the Euler method with a step size of 0.1 to find approximations for the values of x and y when $t = 0.5$ for each of the following systems of coupled differential equations with the given initial conditions:

(a)

$$\frac{dx}{dt} = x^2 + 4ty$$

$$\frac{dy}{dt} = -3x + y - t$$

$$x = 2, \;\; y = 1 \;\; when \;\; t = 0$$

[6 marks]

Question 1b

(b)

$$\dot{x} = -x + e^{-t}y$$

$$\dot{y} = e^{-t}x + y$$

$$x = 1, \;\; y = -1 \;\; when \;\; t = 0$$

[6 marks]

Question 2a

Consider the following system of differential equations:

$$\frac{dx}{dt} = x + 2y$$

$$\frac{dy}{dt} = -3x - 4y$$

(a)

Find the eigenvalues and corresponding eigenvectors of the matrix $\begin{pmatrix} 1 & 2 \\ -3 & -4 \end{pmatrix}$.

[6 marks]

Question 2b

(b)
Hence write down the general solution of the system.

[2 marks]

Question 2c

When $t=0$, $x=-10$ and $y=17$.

(c)
Use the given initial condition to determine the exact solution of the system.

[3 marks]

Question 2d

(d)
By considering appropriate limits as $t \to \infty$, determine the long-term behaviour of the variables x and y.

[2 marks]

Question 3a

The rates of change of two variables, x and y, are described by the following system of differential equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 4x + y$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -5x - 2y$$

The matrix $\begin{pmatrix} 4 & 1 \\ -5 & -2 \end{pmatrix}$ has eigenvalues of 3 and -1 with corresponding eigenvectors $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -5 \end{pmatrix}$. Initially $x = 1$ and $y = 3$.

a)
Use the above information to find the exact solution to the system of differential equations.

[5 marks]

Question 3b

(b)
Use the Euler method with a step size of 0.2 to find approximations for the values of x and y when $t = 1$.

[6 marks]

Question 3c

(c)

(i)

Find the percentage error of the approximations from part (b) compared with the exact values of x and y when $t = 1$.

(ii)

Comment on the accuracy of the approximations in part (b), and explain how they could be improved.

[5 marks]

Question 4a

For each of the general solutions to a system of coupled differential equations given below,

(i)
sketch the phase portrait for the system

(ii)
state whether the point $(0, 0)$ is a stable equilibrium point or an unstable equilibrium point.

(a)

$$\boldsymbol{x} = Ae^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + Be^{2t}\begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

[4 marks]

Question 4b

(b)

$$\boldsymbol{x} = Ae^{-2t}\begin{pmatrix} 4 \\ 1 \end{pmatrix} + Be^{-3t}\begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

[4 marks]

Question 4c

(c)

$$\boldsymbol{x} = Ae^{t}\begin{pmatrix}-1\\1\end{pmatrix} + Be^{-t}\begin{pmatrix}4\\3\end{pmatrix}$$

[4 marks]

Question 5a

The behaviour of two variables, x and y, is modelled by the following system of differential equations:

$$\frac{dx}{dt} = 3x - 5y \qquad \frac{dy}{dt} = x - y$$

where $x = 1$ and $y = 1$ when $t = 0$.

The matrix $\begin{pmatrix}3 & -5\\1 & -1\end{pmatrix}$ has eigenvalues of $1 + i$ and $1 - i$.

(a)

(i)

Find the values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ at the point $(0, 1)$.

(ii)

Hence sketch the phase portrait of the system with the given initial condition.

[4 marks]

Question 5b

It is suggested that the variables might better be described by the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -3x - 5y \quad \frac{\mathrm{d}y}{\mathrm{d}t} = x + y$$

with the same initial conditions.

(b) Calculate the eigen values of the matrix $\begin{pmatrix} -3 & -5 \\ 1 & 1 \end{pmatrix}$

[3 marks]

Question 5c

(c)

Hence describe how your phase portrait from part (a)(ii) would change to represent this new system of differential equations.

[2 marks]

Question 6a

Scientists have been tracking levels, x and y, of two atmospheric pollutants, and recording the levels of each relative to historical baseline figures (so a positive value indicates an amount higher than the baseline and a negative value indicates an amount less than the baseline). Based on known interactions of the pollutants with each other and with other substances in the atmosphere, the scientists propose modelling the situation with the following system of differential equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x - 2y$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = x - y$$

a) Find the values of $\frac{\mathrm{d}x}{\mathrm{d}t}$ and $\frac{\mathrm{d}y}{\mathrm{d}t}$ at the points $(1, 0)$ and $(0, 1)$.

[2 marks]

Question 6b

(b) Find the eigenvalues of the matrix $\begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}$.

[3 marks]

Question 6c

At the start of the study both pollutants are above baseline levels, with $x = 5$ and $y = 3$.

(c)
Use the above information to sketch a phase portrait showing the long-term behaviour of x and y.

[4 marks]

Question 7a

Two types of bacteria, X and Y, are being grown on a culture plate in a research lab. From past studies of the two bacteria and their interactions, the researchers believe that the growth of the two populations may be represented by the following differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -5x + 4y \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -8x + 7y$$

for populations of x thousand and y thousand bacteria of types X and Y respectively. Initially the plate contains $20\,000$ bacteria of type X and $21\,000$ of type Y.

a)

Find the eigenvalues and corresponding eigenvectors of the matrix $\begin{pmatrix} -5 & 4 \\ -8 & 7 \end{pmatrix}$.

[6 marks]

Question 7b

(b) Find the values of $\frac{\mathrm{d}x}{\mathrm{d}t}$ and $\frac{\mathrm{d}y}{\mathrm{d}t}$ when $t = 0$.

[2 marks]

Question 7c

(c)
Sketch a possible trajectory for the growth of the two populations of bacteria, being sure to indicate any asymptotic behaviour.

[4 marks]