1.8 Eigenvalues & Eigenvectors

Question Paper

Course	DP IB Maths
Section	1. Number & Algebra
Topic	1.8 Eigenvalues & Eigenvectors
Difficulty	Very Hard

Time allowed: 110

Score: /90

Percentage: /100

© 2015-2023 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

Question 1a

Find the eigenvalues and corresponding eigenvectors for each of the following matrices:

a)

$$A=\begin{pmatrix} 2 & -\frac{7}{3} \\ \frac{13}{6} & -\frac{5}{2} \end{pmatrix}$$

[6 marks]

Question 1b

b)

$$B=\begin{pmatrix} 0.1 & 0.5 \\ -0.02 & 0.3 \end{pmatrix}$$

[5 marks]

© 2015–2023 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

Question 2a

Find the eigenvalues and corresponding eigenvectors for each of the following matrices:

a)

$$C = \begin{pmatrix} -3 & 17 \\ -2 & 3 \end{pmatrix}$$

[7 marks]

Question 2b

b)

$$D = \begin{pmatrix} 1.5 & -2.5 \\ 4.5 & 2.5 \end{pmatrix}$$

[7 marks]

© 2015-2023 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

Question 3a

Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 matrix with real-valued elements $a,\ b,\ c$ and d. Let λ_1 and λ_2 be the eigenvalues of matrix M.

a)
In the case where $\lambda_1 \neq \lambda_2$, show that

i)
$a + d = \lambda_1 + \lambda_2$

ii)
$\det M = \lambda_1 \lambda_2$

[7 marks]

© 2015-2023 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

Question 3b

b)
In the case where $\lambda_1 = \lambda_2$, show that $(a - d)^2 + 4bc = 0$.

[4 marks]

Question 3c

c)
Hence show that the results of part (a) are also true when matrix ***M*** has a single repeated eigenvalue.

[3 marks]

© 2015-2023 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

Question 4a

Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix with real-valued elements $a,\ b,\ c$ and d which are such that $a + c = 1$ and $b + d = 1$.

a)
Show that the eigenvalues of M are 1 and $(a + d) - 1$.

[5 marks]

Question 4b

b)
In the case where $M \neq I$, find the eigenvectors of M corresponding to the eigenvalues found in part (a). Give your answers, where appropriate, in terms of *a* and *d* only.

[4 marks]

© 2015-2023 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

Question 4c

c)
In the case where $M = I$, describe briefly the eigenvalues and eigenvectors of M.

[3 marks]

Question 5

Consider the matrix M defined as

$$M = \begin{pmatrix} \frac{23}{14} & \frac{5}{7} \\ \frac{15}{14} & k \end{pmatrix}$$

where $k \in \mathbb{R}$ is a constant. It is given that $-\frac{1}{2}$ is an eigenvalue of M.

By first finding the value of k, diagonalise M by writing it in the form PDP^{-1} for appropriate matrices P and D.

[8 marks]

© 2015–2023 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

Question 6a

Consider the matrix $M = \begin{pmatrix} p & 1 \\ 2 & q \end{pmatrix}$, where $p,\ q \in \mathbb{R}$ are constants.

It is given that -6 is an eigenvalue of M, and also that $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector of M which does not correspond to the eigenvalue -6.

a)
By first finding the values of p and q, write M in the form $M = PDP^{-1}$ for appropriate matrices P and D.

[9 marks]

© 2015–2023 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

Question 6b

b)
Hence show that

$$M^n = \frac{(-1)^n}{3}\begin{pmatrix} 3^n + 2\times 6^n & 3^n - 6^n \\ 2(3^n - 6^n) & 2\times 3^n + 6^n \end{pmatrix}$$

[4 marks]

© 2015-2023 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

Question 7a

Two towns, Avaricia and Covetton, are located on opposite sides of a national park. The two towns are heavily dependent on tourism, and they compete with one another both for the business of tourists coming to the park, and for residents to work in the tourism industry.

Government officials studying the two towns indicate the population of Avaricia by a, and the population of Covetton by c. If the respective populations at a particular point in time are a_n and c_n, then data suggest that the populations one year later may be modelled by the following system of coupled equations:

$$a_{n+1} = 1.025a_n - 0.075c_n$$

$$c_{n+1} = -0.025a_n + 0.975c_n$$

Let a_0 and c_0 indicate the respective populations of the two towns at the start of the study.

a)
Use a matrix method to show that the respective populations after n years are predicted by the model to be

$$a_n = 0.75(a_0 - c_0)(1.05^n) + (0.25a_0 + 0.75c_0)(0.95^n)$$
$$c_n = 0.25(c_0 - a_0)(1.05^n) + (0.25a_0 + 0.75c_0)(0.95^n)$$

[11 marks]

© 2015–2023 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

Question 7b

b)

Describe what the model predicts in the long term for the populations of the two towns, for each of the following situations:

i)

$a_0 = c_0$

ii)

$a_0 > c_0$

iii)

$a_0 < c_0$

[7 marks]

© 2015–2023 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

© 2015-2023 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers