# 4.7 Further Probability Distributions

## Question Paper

| | |
|---|---|
| Course | DP IB Maths |
| Section | 4. Statistics & Probability |
| Topic | 4.7 Further Probability Distributions |
| Difficulty | Hard |

**Time allowed:** 90

**Score:** /71

**Percentage:** /100

## Question 1a

A continuous random variable $X$ has the probability density function given by

$$f(x) = \begin{cases} k \sin 2x, & 0 \leq x \leq \frac{\pi}{3} \\ 0, & \text{otherwise} \end{cases}$$

(a)
Find the value of $k$.

**[2 marks]**

## Question 1b

(b)
Giving your answers to three significant figures, find
(i)
the mean of $X$,
(ii)
the mode of $X$.

**[3 marks]**

## Question 1c

(c)
(i)
Write down $\mathrm{P}\left(X = \frac{\pi}{3}\right)$.

(ii) Show that the median, $m$, of $X$ lies in the interval $\frac{\pi}{6} < m < \frac{\pi}{3}$.

**[2 marks]**

## Question 2a

The continuous random variable $X$ has probability density function

$$f(x) = \begin{cases} k(x+4), & 0 \leq x \leq 1 \\ k(6-x), & 1 < x \leq 6 \\ 0, & \text{otherwise} \end{cases}$$

(a)
Find the value of $k$.

**[3 marks]**

## Question 2b

(b)
Sketch the probability density function.

**[2 marks]**

## Question 2c

(c)
Find:

(i)
$\mathrm{E}(X)$,
(ii)
$\mathrm{Var}(X)$,
(iii)
$\mathrm{P}(0.5 \leq X \leq 1.5)$.

**[6 marks]**

## Question 3a

The discrete random variable, $X$, has probability distribution function

$$f(x) = \begin{cases} \dfrac{k}{2x}, & x = 2,4,6,12 \\ 0, & \text{otherwise} \end{cases}$$

(a)
Show that $k = 2$.

**[2 marks]**

## Question 3b

(b)
Find the expected value and variance of $X$.

**[2 marks]**

## Question 3c

As part of a game, a four-sided spinner is created with the numbers 2, 4, 6 and 12. The discrete random variable $X$ is used to model the number that the spinner lands on. The score allocated to a player on their turn is 4 more than double the value the spinner lands on.

(c)
Find the expected value and variance of a player's score from a single spin.

**[2 marks]**

## Question 4a

A UK energy company charges £0.22 per kilowatt hour (kWh) of electricity used.
The amount of energy used per day by the company's customers, $X$ kWh, follows the following probability density function

$$f(x) = \begin{cases} \dfrac{x(k-x)}{972}, & 0 \leq x \leq 18 \\ 0, & \text{otherwise} \end{cases}$$

(a)
Show that $k = 18$.

**[2 marks]**

## Question 4b

(b)
A customer's total daily charge consists of a fixed (standing) charge of £0.38 per day plus the charge for the electricity used.
(i)
Find the expected total daily charge.
(ii)
Find the standard deviation for the total daily charge.

**[6 marks]**

## Question 5a

Consider the function $f$ defined by,

$$f(x) = \begin{cases} k(x-4)(x-5)^2, & 4 \leq x \leq 5 \\ 0, & \text{otherwise} \end{cases}$$

(a)
Use your GDC to verify that $f(x)$ can represent a probability density function for a continuous random variable $X$ in the case when $k = 12$. Explain your verification process in full.

**[2 marks]**

## Question 5b

(b)
Use your GDC to find the mode of $X$.

**[1 mark]**

## Question 5c

(c)
Use your GDC to estimate the median of $X$.

**[2 marks]**

## Question 6a

Consider the probability density function for a continuous random variable, $X$

$$f(x) = \begin{cases} \frac{1}{64}(3x - 20), & a \le x \le 12 \\ \frac{1}{64}(x - 16)^2, & 12 \le x \le 16 \\ 0, & \text{otherwise} \end{cases}$$

(a)

(i) Given that $P(12 \le X \le 16) = \frac{1}{3}$, find the value of $a$.

(ii)
Briefly explain how, without further calculations, it can be deduced that the median of $X$, $m$, lies in the interval $a \le m \le 12$.

**[2 marks]**

## Question 6b

(b)
Find
(i)
$\mathrm{E}(X)$
(ii)
$\mathrm{E}(X^2)$
(iii)
$\mathrm{Var}(X)$.

**[4 marks]**

## Question 7a

In a quick-fire quiz consisting of 25 questions, contestants have just over 2 seconds to answer each question. The time taken on any single question in the quiz is modelled by the continuous random variable, $T$, which has probability density function

$$f(t) = \begin{cases} \sin\frac{1}{2}t, & 0 \le t \le T_{\max} \\ 0, & \text{otherwise} \end{cases}$$

a)
Find the exact value of $T_{\max}$ and verify that this is just over 2 seconds.

**[2 marks]**

## Question 7b

(b)
Find the probability that a contestant takes between 1 and 2 seconds to answer a question.

**[2 marks]**

## Question 7c

(c)
Sketch the graph of $y = f(t)$.

**[1 mark]**

## Question 7d

(d)
Write down the mode of $T$.

**[1 mark]**

## Question 7e

(e)
Find the median of $T$.

**[2 marks]**

## Question 7f

(f)
Find the mean time for a contestant to answer all 25 questions.

**[3 marks]**

## Question 8a

The continuous random variable, $\Theta$, has probability density function

$$f(\theta) = \begin{cases} \frac{1}{p}\cos(\theta - q), & a \leq \theta \leq a + \pi \\ 0, & \text{otherwise} \end{cases}$$

a)
Given that $f(a) = 0$, write down, in terms of $a$, the mean, median and mode of $\Theta$, briefly explaining how you obtained your answers.

**[2 marks]**

## Question 8b

(b) (i) Given that $a = \frac{\pi}{6}$, deduce the smallest positive value of $q$.

(ii) Hence, or otherwise, find the value of $p$.

**[3 marks]**

## Question 8c

(c)
$R$ and $S$ are values of $\Theta$ such that $\mathrm{P}(\Theta < R) = \mathrm{P}(\Theta > S)$.
Write down an equation connecting $R$ and $S$.

**[1 mark]**

## Question 9a

Two random variables, $X$ and $Y$ are such that $\mathrm{E}(3X + 5) = \mathrm{Var}(3Y + 5)$.

It is also known that $\mathrm{E}(Y) = \sqrt{\mathrm{E}(X)}$ and $\mathrm{E}(Y^2) = [\mathrm{E}(X)]^2$.

(a)
Show that $9[\mathrm{E}(X)]^2 - 12\mathrm{E}(X) - 5 = 0$.

**[3 marks]**

## Question 9b

(b)
Given that $\mathrm{E}(X) > 0$, find $\mathrm{E}(X)$.

**[2 marks]**

## Question 10a

Consider the function

$$f(x) = \frac{1}{2\pi}\sqrt{4 - x^2}$$

a)
Sketch the graph of $y = f(x)$.

**[2 marks]**

## Question 10b

(b)
Write down a domain for $f(x)$ such that it could be a probability density function for a continuous random variable $X$.

**[1 mark]**

## Question 10c

(c)
Write down
(i)
$\mathrm{P}(X < -1)$,
(ii)
$\mathrm{E}(X)$,
(iii)
The median of $X$.

**[3 marks]**