5.9 Advanced Integration

Question Paper

Course	DP IB Maths
Section	5. Calculus
Topic	5.9 Advanced Integration
Difficulty	Very Hard

Time allowed: 130

Score: /103

Percentage: /100

Question 1a

Find the following indefinite integrals:

(a)

$$\int 5x^2 \sec^2(2x^3 - 1)\, dx$$

[3 marks]

Question 1b

(b)

$$\int \frac{1}{\sin\frac{x}{4}\tan\frac{x}{4}}\, dx$$

[3 marks]

Question 1c

(c)

$$\int \left(1 + \cos^2\left(\frac{x-\pi}{2}\right)\right)\left(\frac{\tan^2\left(\frac{x-\pi}{2}\right)}{1 - \cos^4\left(\frac{\pi - x}{2}\right)}\right) dx$$

[4 marks]

Question 2a

Find the following indefinite integrals:

(a)

$$\int \frac{1}{12}\left(\ln \frac{1}{64}\right)4^x \, \mathrm{d}x$$

[3 marks]

Question 2b

(b)

$$\int \frac{3}{11+(x-7)^2} \, \mathrm{d}x$$

[3 marks]

Question 2c

(c)

$$\int \frac{e^x}{\sqrt{25 - e^{2x}}} \, \mathrm{d}x$$

[3 marks]

Question 2d

(d)

Using a sketch, briefly describe the family of graphs corresponding to all the possible specific solutions to the integral in part (a).

[3 marks]

Question 3

Show that

$$\int_{-8}^{-5} \frac{x + 15}{x^2 - 3x - 28} \, \mathrm{d}x = 6 \ln 2 - 2 \ln 5$$

[8 marks]

Question 4

Use the substitution $u^2 = e^x - 1$ to find the exact value of the following definite integral:

$$\int_{\ln 2}^{\ln 3} \frac{e^{4x}}{2e^x - 2} \, \mathrm{d}x$$

being sure to simplify your answer as much as possible.

[7 marks]

Question 5

Find the exact value of the definite integral

$$\int_{1}^{3+2\sqrt{3}} \frac{2}{\sqrt{7+6x-x^2}}\,\mathrm{d}x$$

[9 marks]

Question 6

Use the substitution $u = \sqrt{x^2 - 5}$ to find the indefinite integral

$$\int \frac{x}{3 - \sqrt{x^2 - 5}} \, \mathrm{d}x$$

[7 marks]

Question 7a

(a)

Use integration by parts to find the indefinite integral

$$\int \arccos x \, \mathrm{d}x$$

[5 marks]

Question 7b

(b)

Hence find the exact value of the definite integral

$$\int_{-\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \arccos x \, \mathrm{d}x$$

[3 marks]

Question 8a

(a)

Show that

$$\int ax^2 \sin bx \, \mathrm{d}x = \left(\frac{2a}{b^2} - \frac{a}{b}x^2\right)\cos bx + \frac{2a}{b^2}x \sin bx + c$$

where $a, b \in \mathbb{R}$ are non-zero constants, and c is a constant of integration.

[6 marks]

Question 8b

(b)

A continuous random variable X has the probability density function f given by

$$f(x) = \begin{cases} ax^2 \sin\left(\frac{\pi x}{5}\right), & 0 \leq x \leq 5 \\ 0, & \text{otherwise} \end{cases}$$

Find the value of a.

[4 marks]

Question 9

Find the indefinite integral

$$\int e^{3x} \sin 2x \, dx$$

[7 marks]

Question 10a

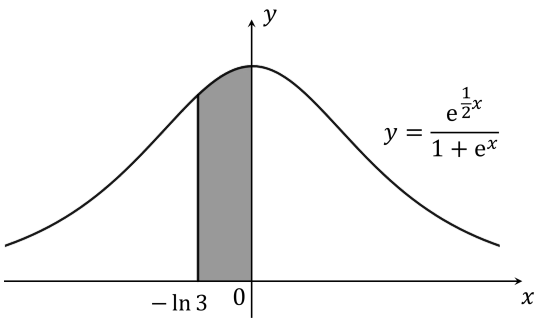
Let f be the function defined by $f(x) = \frac{e^{\frac{1}{2}x}}{1 + e^{x}}$.

(a)
Show that f is an even function.

[2 marks]

Question 10b

The diagram shows a part of the graph of $y = f(x)$. The shaded region is the region bounded by the curve, the positive y-axis, the negative x-axis, and the line $x = -\ln 3$.



(b)
Find the exact area of the shaded region.

[7 marks]

Question 10c

(c)
Given that

$$\int_{-\ln 3}^{a} \frac{e^{\frac{1}{2}x}}{1+e^{x}}\,\mathrm{d}x = \frac{\pi}{3}$$

write down the value of a. Justify your answer.

[2 marks]

Question 11

Consider the curve with equation

$$x = \mathrm{cosec}^{n} y \cot y, \qquad \frac{\pi}{6} \leq y \leq \frac{\pi}{2}$$

where $n \in \mathbb{R}, n \neq 0,$. It is given that, for any allowed value of n, x has a defined non-negative value for all values of y in the stated domain.

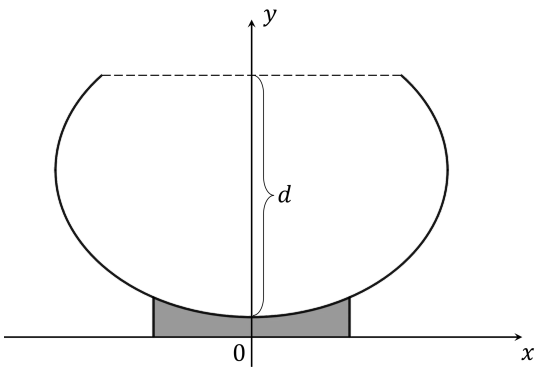
Let A be the area enclosed by the curve, the y-axis, and the lines $y = \frac{\pi}{6}$ and $y = \frac{\pi}{2}$.

Find an expression for A in terms of n.

[6 marks]

Question 12

The diagram below shows the cross-section of a goldfish bowl to be produced by Pieseize Manufacturing, a specialist company supplying products for goldfish enthusiasts.



The glass part of the bowl sits on a solid base, indicated by the shaded region on the diagram. The cross-section of the glass part of the bowl is symmetrical about the y-axis, and may be described by the curve with equation

$$\frac{x^2}{400} + \frac{(y-17)^2}{225} = 1$$

The dashed horizontal line represents the diameter of the open top of the fishbowl. The maximum depth of the fishbowl, measured along the y-axis from the diameter of the open top to where the glass part of the bowl meets the base, is indicated by d in the diagram. All coordinates are expressed in centimetres, and for purposes of answering this question the thickness of the glass sides of the bowl may be regarded as negligible.

The owner of the company, Skodyn Pieseize, is extremely superstitious and is obsessed with the number 23. Therefore he insists that the capacity of the glass part of this new fishbowl must be exactly 23 litres. Find the value of d that satisfies this requirement.

[8 marks]