5.4 Further Integration

Question Paper

Course	DP IB Maths
Section	5. Calculus
Topic	5.4 Further Integration
Difficulty	Very Hard

Time allowed: 120

Score: /95

Percentage: /100

Question 1a

Consider the function f defined by $f(x) = (x^2 - 3x + 2)(x + 2),\ -3 \leq x \leq \dfrac{5}{4}$.

a)
Find the indefinite integral

$$\int (x^2 - 3x + 2)(x + 2)\,dx$$

[3 marks]

Question 1b

b)
Use your answer to part (a) to calculate the area of the region enclosed by the graph of $y = f(x)$ and the x-axis.

[4 marks]

Question 2a

a)
Find the indefinite integral for

$$\int \sin\left(\dfrac{\sqrt{3}}{2}x\right)dx$$

[2 marks]

Question 2b

b)

Find the indefinite integral for

$$\int \frac{7}{e^{4x-9}}\,\mathrm{d}x$$

[2 marks]

Question 2c

c)

Find an expression for y given that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \cos\!\left(2\!\left(\frac{\pi}{8} - x\right)\right)$$

[2 marks]

Question 3a

a)

Find the indefinite integral

$$\int -\frac{7}{5x}\,\mathrm{d}x$$

[2 marks]

Question 3b

b)
Find an expression for given that

$$\frac{dy}{dx} = xe^{x^2-2}$$

and also that $y = 3$ when $x = -\sqrt{2}$

[3 marks]

Question 4a

a)
Find the indefinite integral for

$$\int \left(\sqrt{3x} + \frac{5}{\sqrt[3]{8x}} \right) dx$$

[3 marks]

Question 4b

b)
Find the indefinite integral for

$$\int \frac{(8x)^{\frac{2}{3}} - 5x^{\frac{1}{6}}}{x\sqrt[3]{x}} dx$$

[3 marks]

Question 5a

a)
Find the indefinite integral

$$\int \frac{x + 2x^2}{(1 - x^2)(2 + x^2)}\,dx$$

[4 marks]

Question 5b

b)

Let $g'(x) = \dfrac{\cos(\ln x)}{x}$ for $x > 0$.
Find $g(x)$ given that $g(1) = \pi$.

[5 marks]

Question 5c

c)
Show that

$$\int \tan x \, dx = \ln \left| \frac{1}{\cos x} \right| + c$$

[4 marks]

Question 6

A curve with equation $y = f(x)$ is such that

$$\frac{dy}{dx} = \frac{k}{(1 + \sin \pi x)(1 - \sin \pi x)}$$

where k is a real constant.

Given that the curve passes through the points $(0, -3)$ and $\left(-\frac{1}{4}, -\pi \right)$, find $f(x)$.

[6 marks]

Question 7a

a)

Explain why

$$\frac{1}{\tan \theta} = \frac{\cos \theta}{\sin \theta}$$

[2 marks]

Question 7b

b)

Use definite integration, along with the result from part (a), to show that

$$\int_0^{\sqrt{\frac{\pi}{6}}} \frac{x}{\tan\left(x^2 - \frac{2\pi}{3}\right)} \, dx = -\frac{1}{2} \ln\left(\frac{\sqrt{3}}{2}\right)$$
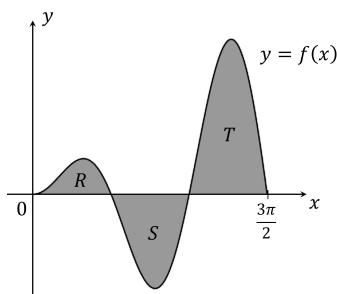
[7 marks]

Question 7c

c)
Using your knowledge of the natural logarithm function, explain (without using your GDC) why the value of the integral found in part (b) is a positive number.

[1 mark]

Question 8a

The diagram below shows the graph of the function f which is defined by

$$f(x) = x\sin 2x, \qquad 0 \le x \le \frac{3\pi}{2}$$

The shaded region in the diagram is the region enclosed by the x-axis and the graph of $y = f(x)$. The three sub-parts of the shaded region are denoted by R, S and T, as shown.

(a)   Find the value of

$$\int_0^{\frac{3\pi}{2}} x \sin 2x \, dx$$

[4 marks]

Question 8b

b)
Find the individual areas of each of the three sub-parts *R*, *S* and *T* of the shaded region.
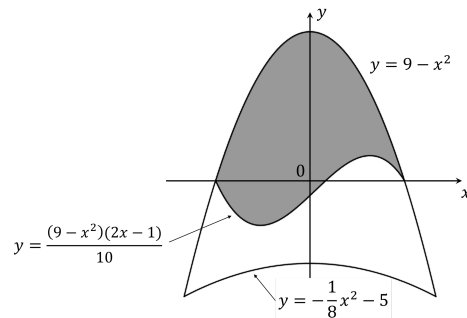
[5 marks]

Question 8c

c)
Compare the sum of the answers in part (b) to the answer in part (a) and comment on the result.

[4 marks]

Question 9

The diagram below depicts the design for a new company logo. The upper border of the logo is formed by a part of the curve with equation $y = 9 - x^2$, while the lower border of the logo is formed by a part of the curve with equation $y = \dfrac{1}{8}x^2 - 5$. As shown in the diagram, the logo is divided into a shaded part and an unshaded part by a part of the curve with equation $y = \dfrac{(9 - x^2)(2x - 1)}{10}$.


Find the percentage of the total area of the logo that is shaded.

[8 marks]

Question 10a

The following diagram shows a part of the graph of the curve with equation $y^2 = k\left(1600 - 25(x-8)^2\right)$, where $k > 0$ is a constant. The point marked A is the vertex of the curve. Region R is the region enclosed by the curve and the x-axis, for the part of the curve where y is non-negative. Region S is the region enclosed by the curve, the positive y-axis, and the line through point A with gradient zero.
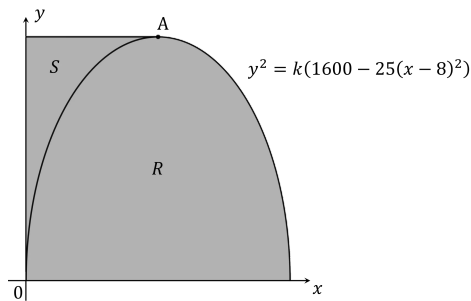
When region R is rotated 2π radians about the x-axis, the resultant solid of revolution has a volume equal to $\dfrac{12800\pi}{3}$ units^3.

a)
Find the area of region S by calculating an area between the curve and the y-axis.
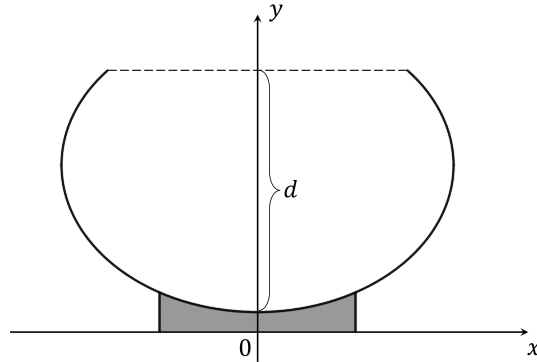
[9 marks]

Question 10b

b)
Find the area of region S by calculating an area between the curve and the x-axis. Confirm that this matches your answer to part (a).

[4 marks]

Question 11

The diagram below shows the cross-section of a goldfish bowl to be produced by Pieseize Manufacturing, a specialist company supplying products for goldfish enthusiasts.

The glass part of the bowl sits on a solid base, indicated by the shaded region on the diagram. The cross-section of the glass part of the bowl is symmetrical about the y-axis, and may be described by the curve with equation

$$\frac{x^2}{400} + \frac{(y-17)^2}{225} = 1$$

The dashed horizontal line represents the diameter of the open top of the fishbowl. The maximum depth of the fishbowl, measured along the y-axis from the diameter of the open top to where the glass part of the bowl meets the base, is indicated by d in the diagram. All coordinates are expressed in centimetres, and for purposes of answering this question the thickness of the glass sides of the bowl may be regarded as negligible.

The owner of the company, Skodyn Pieseize, is extremely superstitious and is obsessed with the number 23. Therefore he insists that the capacity of the glass part of this new fishbowl must be exactly 23 litres. Find the value of d that satisfies this requirement.

[8 marks]