1.8 Eigenvalues & Eigenvectors

Question Paper

Course	DP IB Maths
Section	1. Number & Algebra
Topic	1.8 Eigenvalues & Eigenvectors
Difficulty	Medium

Time allowed: 80

Score: /65

Percentage: /100

Question 1a

Consider the 2×2 matrix $\boldsymbol{A}$ defined by

$$\boldsymbol{A} = \begin{pmatrix} 0.1 & 0.4 \\ 0.9 & 0.6 \end{pmatrix}$$

(a)

(i)

Find the characteristic polynomial of $\boldsymbol{A}$.

(ii)

By solving an appropriate equation with the characteristic polynomial, find the eigenvalues λ_1 and λ_2 of $\boldsymbol{A}$.

[3 marks]

Question 1b

Let $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ be the eigenvectors of $\boldsymbol{A}$ corresponding to λ_1 and λ_2 respectively.

(b)

By solving the eigenvector equations $\boldsymbol{A}\boldsymbol{x}_1 = \lambda_1 \boldsymbol{x}_1$ and $\boldsymbol{A}\boldsymbol{x}_2 = \lambda_2 \boldsymbol{x}_2$, find eigenvectors $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$.

[4 marks]

Question 1c

(c) Show that the answers to part (b) could alternatively have been found by solving the equations $(\boldsymbol{A} - \lambda_1 \boldsymbol{I})\,\boldsymbol{x}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $(\boldsymbol{A} - \lambda_2 \boldsymbol{I})\,\boldsymbol{x}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, where $\boldsymbol{I}$ is the 2×2 identity matrix.

[3 marks]

Question 2

Find the eigenvalues and corresponding eigenvectors for the matrix $\boldsymbol{A}$ defined as

$$\boldsymbol{A} = \begin{pmatrix} -1 & 4 \\ 1 & 2 \end{pmatrix}$$

[1 mark]

Question 3

Consider the matrix $\boldsymbol{B}$ defined as

$$\boldsymbol{B} = \begin{pmatrix} 4 & -6 \\ 1 & -2 \end{pmatrix}$$

Find the eigenvalues and corresponding eigenvectors of $\boldsymbol{B}$.

[6 marks]

Question 4a

Find the eigenvalues for each of the following matrices:

(a)

$$\boldsymbol{C} = \begin{pmatrix} -2 & 13 \\ -1 & 2 \end{pmatrix}$$

[3 marks]

Question 4b

(b)

$$\boldsymbol{D} = \begin{pmatrix} 6 & -1 \\ 17 & -2 \end{pmatrix}$$

[3 marks]

Question 5a

Consider the matrix $\boldsymbol{M}$ defined as

$$\boldsymbol{M} = \begin{pmatrix} -1 & k \\ 3 & -1 \end{pmatrix}$$

where $k \in \mathbb{R}$ is a constant.

The eigenvalues of $\boldsymbol{M}$ are 2 and -4.

(a)
Find the value of k.

[3 marks]

Question 5b

(b)
Find the eigenvectors of $\boldsymbol{M}$ that correspond to the two eigenvalues.

[3 marks]

Question 5c

(c)
Hence write $\boldsymbol{M}$ in the form $\boldsymbol{PDP}^{-1}$, where $\boldsymbol{P}$ is a matrix of eigenvectors and $\boldsymbol{D}$ is a diagonal matrix of eigenvalues.

[2 marks]

Question 6a

(a)
It is given that, for $n \times n$ matrices $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$,

$$\boldsymbol{A} = \boldsymbol{BCB}^{-1}$$

Use the properties of matrices and matrix inverses to show that $\boldsymbol{A}^2 = \boldsymbol{BC}^2\boldsymbol{B}^{-1}$.

[3 marks]

Question 6b

Consider the matrix $\boldsymbol{M} = \begin{pmatrix} 3 & -2 \\ p & 1 \end{pmatrix}$, where $p \in \mathbb{R}$ is a constant and where it is given that $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector of $\boldsymbol{M}$.

(b)
Find the value of p.

[3 marks]

Question 6c

(c)
Hence, by first finding the eigenvalues and the other eigenvector of $\boldsymbol{M}$, write $\boldsymbol{M}$ in the form $\boldsymbol{M} = \boldsymbol{PDP}^{-1}$ for appropriate matrices $\boldsymbol{P}$ and $\boldsymbol{D}$.

[5 marks]

Question 6d

(d) (i) Use the result of part (c) to show that

$$\boldsymbol{M}^n = \frac{1}{3}\begin{pmatrix} 2(5^n)+(-1)^n & -5^n+(-1)^n \\ -2(5^n)+2(-1)^n & 5^n+2(-1)^n \end{pmatrix}$$

(ii)

Show that the expression for $\boldsymbol{M}^{\mathbf{n}}$ in part (d)(i) gives the expected result when $n = 1$.

[4 marks]

Question 7a

Exobiologists are studying two species of animals in a region of the distant planet Dirion. In the researchers' models the population of Heliors (a predator species) is indicated by h, while the population of Sklyveths (a competing predator species) is indicated by s.

If the respective populations at a particular point in time are h_n and s_n, then the researchers' data suggest that the populations one year later may be given by the following system of coupled equations:

$$h_{n+1} = 1.06h_n - 0.16s_n$$

$$s_{n+1} = -0.04h_n + 0.94s_n$$

(a)
Represent the system of equations in the matrix form $\boldsymbol{x}_{n+1} = \boldsymbol{M}\boldsymbol{x}_n$.

[2 marks]

Question 7b

At the start of the study, there are 600 Heliors and 500 Sklyveths in the region.

(b)
Find the expected size of the respective populations after one year.

[2 marks]

Question 7c

(c)
By first finding the eigenvalues and corresponding eigenvectors of $\boldsymbol{M}$ write $\boldsymbol{M}$ in the form $\boldsymbol{PDP}^{-1}$, where $\boldsymbol{P}$ is a matrix of eigenvectors and $\boldsymbol{D}$ is a diagonal matrix of eigenvalues.

[8 marks]

Question 7d

(d)
Hence show that the respective populations after n years are predicted by the model to be $h_n = 520(0.9^n) + 80(1.1^n)$ and $s_n = 520(0.9^n) - 20(1.1^n)$.

[3 marks]

Question 7e

(e)
Describe what the model predicts in the long term for the populations of the two species, and offer one criticism of the model based on this prediction.

[4 marks]