

1.8 Eigenvalues & Eigenvectors

Question Paper

Course	DPIB Maths
Section	1. Number & Algebra
Topic	1.8 Eigenvalues & Eigenvectors
Difficulty	Hard

Time allowed: 100
Score: /78
Percentage: /100

Question 1a

Consider the 2×2 matrix \mathbf{A} defined by

$$\mathbf{A} = \begin{pmatrix} 0.35 & 0.15 \\ 0.65 & 0.85 \end{pmatrix}$$

a)
(i) Find the characteristic polynomial of \mathbf{A} .

(ii)
Find the eigenvalues of \mathbf{A} .

[3 marks]

Question 1b

Let λ_1 and λ_2 be the eigenvalues found in part (a)(ii), and let \mathbf{x}_1 and \mathbf{x}_2 be the eigenvectors of \mathbf{A} corresponding to λ_1 and λ_2 respectively.

b)
Find eigenvectors \mathbf{x}_1 and \mathbf{x}_2 .

[4 marks]

Question 1c

c)

Explain with justification whether the answers found in part (b) are unique.

[2 marks]**Question 2a**Consider the matrix \mathbf{A} defined as

$$\mathbf{A} = \begin{pmatrix} -1 & 0.75 \\ 4 & -1.5 \end{pmatrix}$$

a)

Find the eigenvalues and corresponding eigenvectors of matrix \mathbf{A} .**[6 marks]**

Question 2b

Now consider the matrix $k\mathbf{A}$ defined as

$$k\mathbf{A} = \begin{pmatrix} -k & 0.75k \\ 4k & -1.5k \end{pmatrix}$$

where $k \neq 0$ is a real constant.

b)
Show that the eigenvectors found in part (a) are also eigenvectors of matrix $k\mathbf{A}$ and determine their corresponding eigenvalues.

[5 marks]

Question 3

Consider the matrix \mathbf{B} defined as

$$\mathbf{B} = \begin{pmatrix} 6 & -2 \\ 1 & 2 \end{pmatrix}$$

Find the eigenvalues and corresponding eigenvectors of \mathbf{B} .

[6 marks]

Question 4a

Find the eigenvalues for each of the following matrices:

a)

$$C = \begin{pmatrix} 5 & -4 \\ 4.5 & -1 \end{pmatrix}$$

[3 marks]**Question 4b**

b)

$$D = \begin{pmatrix} 5k & -4k \\ 8.5k & -5k \end{pmatrix}$$

where $k \neq 0$ is a real constant.**[4 marks]**

Question 4c

c)

Show that $\begin{pmatrix} 2+2i \\ 3 \end{pmatrix}$ is an eigenvector of matrix \mathbf{C} , and find the other eigenvector.

[4 marks]**Question 5a**

Consider the matrix \mathbf{M} defined as

$$\mathbf{M} = \begin{pmatrix} -3 & k \\ 2 & 6 \end{pmatrix}$$

where $k \in \mathbb{R}$ is a constant.

Given that -2 is an eigenvalue of \mathbf{M} ,

a)

find the remaining eigenvalue of \mathbf{M} , as well as the eigenvectors that correspond to the two eigenvalues.

[6 marks]

Question 5b

b)

Hence diagonalise \mathbf{M} by writing it in the form \mathbf{PDP}^{-1} for appropriate matrices \mathbf{P} and \mathbf{D} .

[2 marks]

Question 6a

a)

It is given that, for $n \times n$ matrices \mathbf{A} , \mathbf{B} and \mathbf{C} ,

$$\mathbf{A} = \mathbf{BCB}^{-1}$$

Use the properties of matrices and matrix inverses to explain why $\mathbf{A}^n = \mathbf{BC}^n \mathbf{B}^{-1}$.

[4 marks]

Question 6b

Consider the matrix $\mathbf{M} = \begin{pmatrix} p & 1 \\ -2 & -3 \end{pmatrix}$, where $p \in \mathbb{R}$ is a constant and where it is given that $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of \mathbf{M} .

b)

By first finding the eigenvalues and the other eigenvector of \mathbf{M} , write \mathbf{M} in the form $\mathbf{M} = \mathbf{PDP}^{-1}$ for appropriate matrices \mathbf{P} and \mathbf{D} .

[7 marks]**Question 6c**

c)

(i)

Use the result of part (b) to show that

$$\mathbf{M}^n = (-1)^n \begin{pmatrix} 2 \times 5^n & 4^n - 5^n \\ 2(5^n - 4^n) & 2 \times 4^n - 5^n \end{pmatrix}$$

(ii) Show that the expression for \mathbf{M}^n in part (c)(i) gives the expected result when $n = 3$.

[5 marks]

Question 7a

Exobiologists are studying two species of animals in a region of the distant planet Dirion. In the researchers' models the population of Reddors (a prey species) is indicated by r , while the population of Sklyveths (a predator species that preys on Reddors) is indicated by s .

If the respective populations at a particular point in time are r_n and s_n , then the researchers' data suggest that the populations one year later may be modelled by the following system of coupled equations:

$$r_{n+1} = 1.3r_n - 0.25s_n$$

$$s_{n+1} = 0.07r_n + 0.9s_n$$

a)

Represent the system of equations in the matrix form $\mathbf{x}_{n+1} = \mathbf{M}\mathbf{x}_n$.

[2 marks]

Question 7b

b)

By first finding the eigenvalues and eigenvectors of \mathbf{M} , write \mathbf{M} in the form $\mathbf{M} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$ for appropriate matrices \mathbf{P} and \mathbf{D} .

[8 marks]

Question 7c

At the start of the study there were 2100 Reddors and 2850 Sklyveths in the region.

c)

Show that the respective populations after n years are predicted by the model to be $r_n = 75(1.25^n) + 2025(0.95^n)$ and $s_n = 15(1.25^n) + 2835(0.95^n)$.

[3 marks]

Question 7d

d)

(i)

Determine the ratio of Reddors to Sklyveths that the model predicts will be in the region in the long term. Be sure to justify your answer.

(ii)

Determine the number of years it will take after the start of the study for the population of Reddors to exceed the population of Sklyveths.

[4 marks]