5.2 Further Differentiation

Question Paper

Course	DP IB Maths
Section	5. Calculus
Topic	5.2 Further Differentiation
Difficulty	Very Hard

Time allowed: 110

Score: /89

Percentage: /100

Question 1a

Find an expression for the derivative of each of the following functions:

(a) $f(x) = (12x^2 - 7)\mathrm{e}^{-2x}$

[2 marks]

Question 1b

(b) $g(x) = \frac{\cos 3x}{4-5x^3}$

[3 marks]

Question 1c

(c) $h(x) = (\ln(2x^2 - x - 2))^5$

[3 marks]

Question 2a

Find an expression for the derivative of each of the following functions:

(a) $f(x) = (3x - 1)\mathrm{e}^{\sin x}$

[3 marks]

Question 2b

(b) $g(x) = \ln(\cos(x^2 - 1))$

[3 marks]

Question 2c

(c) $h(x) = \dfrac{-\sin(\mathrm{e}^{-x})}{\mathrm{e}^{x\cos x}}$

[4 marks]

Question 3

Consider the function f defined by $f(x) = -x + \frac{2}{3}\sin^3 x,\ x \in \mathbb{R}$.

Show that f is decreasing everywhere on its domain.

[7 marks]

Question 4a

Consider the function g defined by $g(x) = e^{2x} - 2x,\ x \in \mathbb{R}$.

Point A is the point on the graph of g for which the x-coordinate is $\ln \sqrt{3}$.

(a) Find the equation of the tangent to the graph of g at point A.

[4 marks]

Question 4b

Point B is the point on the graph of g at which the normal to the graph is vertical.

(b) Show that the coordinates of the point of intersection between the tangent to the graph of g at point A and the tangent to the graph of g at point B are

$$\left(\frac{3\ln 3 - 2}{4},\ 1\right)$$

[5 marks]

Question 5

Consider the function h defined by $h(x) = \sin 3x + e^{3\sqrt{3}x} \cos 3x,\ x \in \mathbb{R}$.

Show that the normal line to the graph of h at $x = \frac{\pi}{9}$ intercepts the y-axis at the point

$$\left(0,\ \frac{2\pi}{27} + \frac{\sqrt{3} + e^{\frac{\pi\sqrt{3}}{3}}}{2}\right)$$

[9 marks]

Question 6

Let $f(x) = g(x)h(x)$, where g and h are real-valued functions such that

$$g(x) = \ln\left(\frac{x}{3}\right)h(x)$$

for all $x > 0$.

Given that $h(3) = a$ and $h'(3) = b$, where $a \neq 0$, find the distance between the y-intercept of the tangent to the graph of f at $x = 3$ and the y-intercept of the normal to the graph of f at $x = 3$. Give your answer in terms of a and/or b as appropriate.

[8 marks]

Question 7a

Consider the function f defined by $f(x) = \cos(kx)\,\mathrm{e}^{\sin(kx)}$, $x \in \mathbb{R}$, where $k \neq 0$ is a positive integer.

(a) For the case where $k = 1$, find the number of points in the interval $-\frac{\pi}{2} \leq x < \frac{3\pi}{2}$ at which the graph of f has a horizontal tangent.

[1 mark]

Question 7b

(b) (i) Show algebraically that in general the x-coordinates of the points at which the graph of f has horizontal tangents will be the solutions to the equation

$$\sin^2(kx) + \sin(kx) - 1 = 0$$

(ii) Hence, for the case where $k = 1$, find the x-coordinates of the points identified in part (a).

[7 marks]

Question 7c

(c) (i) Show algebraically that in general the x-coordinates of the points at which the graph of f is neither concave up nor concave down will be the solutions to the equation

$$\sin(2kx) = 0$$

(ii) Hence, for the case where $k = 1$, find the x-coordinates of the points of the points of inflection on the graph of f in the interval $-\frac{\pi}{2} \leq x < \frac{3\pi}{2}$.

[8 marks]

Question 7d

(d) In terms of k, state in general how many (i) turning points and (ii) points of inflection the graph of f will have in the interval $-\frac{\pi}{2} \leq x < \frac{3\pi}{2}$. Give a reason for your answers.

[2 marks]

Question 8

Let $f(x) = \frac{g(x)}{h(x)}$, where g and h are well-defined functions with $h(x) \neq 0$ anywhere on their common domain.

By first writing $f(x) = g(x)[h(x)]^{-1}$, use the product and chain rules to show that

$$f'(x) = \frac{h(x)g'(x) - g(x)h'(x)}{[h(x)]^2}$$

[4 marks]

Question 9a

Consider the function f defined by $f(x) = e^{x^k}$, $x \in \mathbb{R}$, where $k \geq 1$ is a positive integer.

(a) Show that the graph of f will have no points of inflection in the case where $k = 1$.

[2 marks]

Question 9b

(b) Show that, for $k \geq 2$, the second derivative of f is given by

$$f''(x) = kx^{k-2}(kx^k + k - 1)e^{x^k}$$

[5 marks]

Question 9c

(c) Explain why, for $k \geq 2$,

$$-1 < \sqrt[k]{-\frac{k-1}{k}} < -\frac{1}{2}$$

[2 marks]

Question 9d

(d) Hence show that the graph of f will only have points of inflection in the case where k is an odd integer greater than or equal to 3. In that case, give the exact coordinates of the points of inflection, giving your answer in terms of k where appropriate.

[7 marks]