2.6 Further Modelling with Functions

Question Paper

Course	DP IB Maths
Section	2. Functions
Topic	2.6 Further Modelling with Functions
Difficulty	Very Hard

Time allowed: 110

Score: /84

Percentage: /100

Question 1a

If a substance decays by $k\%$ per year, its half-life, H years, is given by

$$H(k) = \frac{\ln 0.5}{\ln\left(1 - \frac{k}{100}\right)}, \qquad 0 < k < 100$$

a)
Find the half-life of a substance with an annual percentage decay rate of 12.2%.

[2 marks]

Question 1b

b)
Find the annual percentage decay rate of a substance with a half-life of 32.5 years.

[2 marks]

Question 1c

Americium-241 has a half-life of 432.2 years and a sample initially contains 40 grams.

c)
Find the number of grams in the sample after 55 years.

[3 marks]

Question 1d

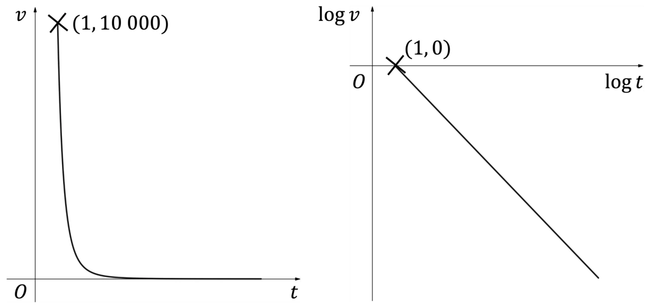
[1 mark]

Question 2a

a)

Consider the function v, where v is a function of t and $1 \le t \le 10.$

Sketches of the graphs of v against t and $\log\ v$ against $\log\ t$, are shown below.



Use the information given in the graph sketches to find v in terms of t.

[3 marks]

Question 2b

b)

The number of calculations, C, a supercomputer can perform per second is related to the speed of its processor, s GHz.

The graph of C against s is an exponential curve passing through the point (4.2×10^8).

The semi-log graph of $\log\ C$ against s is a straight line passing through the point $(3,8)$.

Find C, as a function of s.

[3 marks]

Question 3a

Kerry is organising her local town's summer festival and wants to hire a bouncy castle for a minimum of 2 days up to a maximum of 12 days, depending on cost. She has two potential suppliers to choose from.

Bounce-o-rama can supply a bouncy castle at $£110$ per day for the first three days, $£85$ per day for the next three days and £60 for every day thereafter.

Inflate-o-castle can supply a bouncy castle at $£C$ per day, determined by the equation $C = 163d - 8d^2 - 87$, where d is the number of days the bouncy castle is hired for, with a limit of 10 days.

a)
Given that Kerry has a budget of between £300 and £750, analyse which supplier she should use across the range of her budget in order to obtain the best value for money

[4 marks]

Question 3b

b)
State the number of whole days for which both suppliers charge the same amount.

[1 mark]

Question 3c

c)
Briefly explain why it would not be in Inflate-o-castle's interest to offer more than 10 days of bouncy castle hire using the given cost equation.

[2 marks]

Question 4a

A company makes a device that can be attached to small items such that if the item were to be dropped into water, the item would float rather than sink. In testing the device, an employee throws the device over the edge of a boat and the height, h m, of the device above the level of calm water t seconds after being thrown is modelled by the function

$$h(t) = Ae^{-0.2t}\sin\left(2t + \frac{\pi}{6}\right) \qquad t \geq 0$$

where A is a constant.

a)
Given that the employee throws the keys from a height of above the level of calm water, find the value of *A*.

[2 marks]

Question 4b

b)
Find

(i) the maximum height above the calm water level the device reaches,

(ii) the time at which the device first reaches calm water level,

(iii) the maximum depth below the calm water level the device reaches.

[3 marks]

Question 4c

c)
Find the times between which the device is deeper than 0.5 m below calm water level.

[3 marks]

Question 4d

d)
Describe the motion of the device for large values of t.

[1 mark]

Question 5a

A solid substance, with an extremely high melting point temperature, has a volume of $5\ \text{cm}^3$ at $0°C$.

The substance expands as it is heated, but has a carrying capacity of $4000\ \text{cm}^3$. The volume of the substance is 0.0001 m^3 when heated to $46.5°C$.

The volume, $V\ \text{cm}^3$ at temperature $\theta°C$, of the substance is modelled by the function

$$V = \frac{A}{1 + Be^{-k\theta}}$$

where $A,\ B$ and k are positive constants.

a)
Find the values of $A,\ B$ and k, giving the value of k to two significant figures.

[4 marks]

Question 5b

b)
Find the volume, to the nearest whole cubic centimetre, of the substance when it is heated to $80°C$.

[2 marks]

Question 5c

c)
Briefly explain the relevance of the substance's extremely high melting point temperature in the context of the question.

[1 mark]

Question 6a

The value, V, measured in thousands of dollars, of a mobile phone initially worth $1000, is modelled by the function

$$V = a + b\ln(y+1) \qquad 0 \le y \le c$$

where the age of the mobile phone is y years and a, b and c are constants.

(a) Write down the value of a.

[1 mark]

Question 6b

b)
(i)
Explain what the constant c represents in the context of the model.

(ii)
Given that $b = -\frac{k}{10}$, where k is a positive integer, and that $11.1 < c < 11.2$, find the exact value of b.

[3 marks]

Question 6c

c)
Find the age of the phone when its value has halved from its initial worth.

[2 marks]

Question 6d

d)
State an assumption the model makes about the mobile phone.

[1 mark]

Question 7a

In a fairground "lift and drop" ride, a row of seated passengers are projected vertically upwards until the ride reaches its maximum height ("the lift"), then the ride returns to its starting position ("the drop").

The height of a passenger, h cm, above ground level at time t seconds after the ride begins is modelled by the function

$$h(t) = a\cos(b(t - c)) + d$$

where $a,\ b,\ c$ and d are constants.

(a) Given that

- passengers are 0.75 m above the ground when they board the ride,
- the maximum height a passenger reaches on the ride is 12.25 m,
- it takes 24 seconds for the ride to complete one "lift and drop",
- c takes the smallest positive value possible,

find the values of $a,\ b,\ c$ and d.

[5 marks]

Question 7b

b)
(i)
Find the times at which a passenger is 6 m further from the ground than when they first sat on the ride.
(ii)
Find the height of a passenger from the ground after 10 seconds.

[3 marks]

Question 8a

A temporary lamppost that lights part of a forest pathway is powered via an outdoor electricity generator. The cost, £ C of running the generator for h hours is modelled by the equation

$$C = a\log(bh + 1) \qquad h \geq 0$$

where a and b are positive constants.

a)
Given that, to three significant figures, the cost of running the generator for 2.31 hours is the half the cost of running the generator for 7.28 hours, find the value of b correct to one decimal place.

[3 marks]

Question 8b

b)
Given that is an integer and that £1 would run the generator for just under 20 minutes, find the value of a.

[2 marks]

Question 8c

c)
Find

(i) the cost of running the generator for 12 hours, to the nearest penny (£0.01),

(ii) the maximum length of time (in hours and minutes) the generator can be (continuously) run for at a maximum cost of £8.

[4 marks]

Question 9a

A model of the form $P = P_0 e^{-kt}$ is proposed to model the amount of a toxic gas, *P* parts per million (ppm), in a chamber, *t* seconds after a chemical reaction has taken place inside the chamber. P_0 and k are positive constants.

a)
(i)
Write down the amount of toxic gas in the chamber at the instant the chemical reaction completes?
(ii)
What can you deduce about the background level of the toxic gas inside the chamber? Explain your answer.

[3 marks]

Question 9b

b)
Write down, in terms of k, the exact half-life, $t_{0.5}$ of the toxic gas.

[1 mark]

Question 9c

c)
Given that the amount of toxic gas reduces by 15% every 20 seconds, find its half-life to the nearest second.

[3 marks]

Question 9d

The chemical reaction also produces a harmless gas which has some background level inside the chamber. The amount of this gas increases with time after the chemical reaction has taken place.

d)
Suggest a model for the amount of harmless gas in the chamber t seconds after the chemical reaction completes, defining any constants used and any restrictions on their values.

[2 marks]

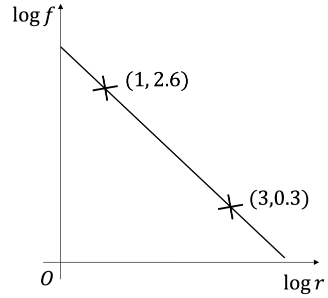
Question 10a

The Theoretical Zipf Distribution relates the rank and frequency of words used in the English language.

For a large body of text a word of rank r, appears f times where $f = kr^{-1}$, k is the number of times the most used word occurs.

The graph below shows $\log f$ plotted against $\log r$ for a particular novel in which the most used word is “the” and occurs 5623 times.

The graph passes through the points $(1, 2.6)$ and $(3,0.3)$.



a)
Use the graph to find f in terms of r and comment on whether the distribution of words in this novel follows the Theoretical Zipf Distribution.

[4 marks]

Question 10b

b)
The 8th ranked word in the novel was “was”. Use your answer to part (a) to estimate how many times “was” was used in the novel.

[2 marks]

Question 11a

The profile of a mountain is modelled by the piecewise function

$$h(x)=\begin{cases}0.02x^2 & 0\le x\le k\\ 0.01(x-16)^3+3.52 & k<x\le 20\end{cases}$$

where h km is the height of the mountain at a horizontal distance x km along the ground from its basecamp.

$h(x)$ is a continuous function at the point $x=k$.

a)
Find the value of k and the height of the mountain at this point.

[3 marks]

Question 11b

b)
Find
(i)
the height of the mountain at a horizontal distance of 6 km along the ground from its basecamp,
(ii)
the horizontal distance along the ground from the basecamp at the point where the mountain's height is 3.82 km.

[2 marks]

Question 11c

c)
Find the height and the horizontal distance from mountain's basecamp at the point where the slope of the mountain is flat.

[3 marks]