IB Maths DP

YOUR NOTES

1. Number & Algebra

CONTENTS

1.1 Number Toolkit

1.1.1 Standard Form

YOUR NOTES

Standard Form

Standard form (sometimes called **scientific notation** or **standard index form**) gives us a way of writing very big and very small numbers using powers of 10.

Why use standard form?

- Some numbers are too big or too small to write easily or for your calculator to display at all
  - Imagine the number 50^{50}, the answer would take 84 digits to write out
  - Try typing 50^{50} into your calculator, you will see it displayed in **standard form**
- Writing very big or very small numbers in standard form allows us to:
  - Write them more neatly
  - Compare them more easily
  - Carry out calculations more easily
- Exam questions could ask for your answer to be written in standard form

How is standard form written?

- In standard form numbers are always written in the form $a \times 10^k$ where a and k satisfy the following conditions:
  - $1 \leq a < 10$
    - So there is one non – zero digit before the decimal point
  - $k \in \mathbb{Z}$
    - So k must be an integer
  - $k > 0$ for large numbers
    - How many times a is multiplied by 10
  - $k < 0$ for small numbers
    - How many times a is divided by 10

How are calculations carried out with standard form?

- Your GDC will display large and small numbers in standard form when it is in normal mode
  - Your GDC may display standard form as *aEn*
    - For example, 2.1×10^{-5} will be displayed as $2.1\text{E}-5$
    - If so, be careful to **rewrite the answer given in the correct form**, you will not get marks for copying directly from your GDC
- Your GDC will be able to carry out calculations in standard form
  - If you put your GDC into scientific mode it will automatically convert numbers into standard form
    - Beware that your GDC may have more than one mode when in scientific mode
    - This relates to the number of significant figures the answer will be displayed in
    - Your GDC may add extra zeros to fill spaces if working with a high number of significant figures, you do not need to write these in your answer

- To add or subtract numbers written in the form $a \times 10^k$ without your GDC you will need to write them in full form first
- To multiply or divide numbers written in the form $a \times 10^k$ without your GDC you can either write them in full form first or use the laws of indices

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Exam Tip

- Your GDC will give very big or very small answers in standard form and will have a setting which will allow you to carry out calculations in scientific notation
- Make sure you are familiar with the form that your GDC gives answers in as it may be different to the form you are required to use in the exam

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Worked Example

Calculate the following, giving your answer in the form $a \times 10^k$, where $1 \le a < 10$ and $k \in \mathbb{Z}$.

i)
3780×200

Using GDC: Choose scientific mode.

Input directly into GDC as ordinary numbers.

$3780 \times 200 = 7.56 \times 10^5$

GDC will automatically give answer in standard form.

Without GDC:

Calculate the value:

$3780 \times 200 = 756000$

Convert to standard form:

$756000 = 7.56 \times 10^5$

$$\boxed{7.56 \times 10^5}$$

ii) $(7 \times 10^5) - (5 \times 10^4)$

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Using GDC: Choose scientific mode.

Input directly into GDC

$7 \times 10^5 - 5 \times 10^4 = 6.5 \times 10^5$

This may be displayed as 6.5E5

Without GDC:

Convert to ordinary numbers:

$7 \times 10^5 = 700\,000$

$5 \times 10^4 = 50\,000$

Carry out the calculation:

$700\,000 - 50000 = 650000$

Convert to standard form:

$650000 = 6.5 \times 10^5$

$$\boxed{6.5 \times 10^5}$$

iii)

$(3.6 \times 10^{-3})(1.1 \times 10^{-5})$

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Input directly into GDC:
(Choose scientific mode).

$(3.6 \times 10^{-3})(1.1 \times 10^{-5}) = 3.96 \times 10^{-8}$

$\boxed{3.96 \times 10^{-8}}$

Note:

$10^{-3} \times 10^{-5} = 10^{-8}$

$(3.6 \times 10^{-3})(1.1 \times 10^{-5}) = 3.96 \times 10^{-8}$

$3.6 \times 1.1 = 3.96$

1.1.2 Laws of Indices

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Laws of Indices

What are the laws of indices?

- Laws of indices (or index laws) allow you to simplify and manipulate expressions involving exponents
  - An exponent is a power that a number (called the base) is raised to
  - Laws of indices can be used when the numbers are written with the same base
- The index laws you need to know are:
  - $(xy)^m = x^m y^m$
  - $\left(\frac{x}{y}\right)^m = \frac{x^m}{y^m}$
  - $x^m \times x^n = x^{m+n}$
  - $x^m \div x^n = x^{m-n}$
  - $(x^m)^n = x^{mn}$
  - $x^1 = x$
  - $x^0 = 1$
  - $\frac{1}{x^m} = x^{-m}$
  - $x^{\frac{1}{n}} = \sqrt[n]{x}$
  - $x^{\frac{m}{n}} = \sqrt[n]{x^m}$
- These laws are **not in the formula booklet** so you must remember them

How are laws of indices used?

- You will need to be able to carry out multiple calculations with the laws of indices
  - Take your time and apply each law individually
  - Work with numbers first and then with algebra
- Index laws only work with terms that have the same base, make sure you **change the base** of the term before using any of the index laws
  - Changing the base means rewriting the number as an exponent with the base you need
  - For example, $9^4 = (3^2)^4 = 3^{2\times4} = 3^8$
  - Using the above can them help with problems like $9^4 \div 3^7 = 3^8 \div 3^7 = 3^1 = 3$

Exam Tip

- Index laws are rarely a question on their own in the exam but are often needed to help you solve other problems, especially when working with logarithms or polynomials
- Look out for times when the laws of indices can be applied to help you solve a problem algebraically

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Worked Example

Simplify the following equations:

i)
$$\frac{(3x^2)(2x^3y^2)}{(6x^2y)}.$$

Apply each law separately :

$$\frac{(\overbrace{3x^2)(2}^{3\times 2=6}x^3y^2)}{6x^2y}$$

expand numerator

$$\frac{(6\overbrace{x^2)(x^3}^{x^2\times x^3 = x^5}y^2)}{6x^2y}$$

cancelling

$$\frac{\cancel{6}x^5y^2}{\cancel{6}x^2y}$$

$x^5 \div x^2 = x^{5-2} = x^3$
$y^2 \div y = y^{2-1} = y$

$$x^3y$$

$$\boxed{\frac{(3x^2)(2x^3y^2)}{6x^2y} = x^3y}$$

ii)
$$(4x^2y^{-4})^3(2x^3y^{-1})^{-2}.$$

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$$(4x^2y^{-4})^3(2x^3y^{-1})^{-2}$$

Rewrite as a fraction

$$\frac{(4x^2y^{-4})^3}{(2x^3y^{-1})^2}$$

expand numerator and denominator

$$\frac{64x^6y^{-12}}{4x^6y^{-2}}$$

cancelling

$$\frac{\cancel{64}\,\cancel{x^6}\,y^2}{\cancel{4}\,\cancel{x^6}\,y^{12}}$$

The negative exponents can be rewritten as their reciprocals

$$16y^{-10}$$

$$\boxed{\frac{16}{y^{10}}}$$

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1.2 Exponentials & Logs

1.2.1 Introduction to Logarithms

Introduction to Logarithms

What are logarithms?

- A logarithm is the inverse of an exponent
  - If $a^x = b$ then $\log_a(b) = x$ where a > 0, b > 0, a ≠ 1
    - This is in the formula booklet
    - Your GDC will be able to use this function to solve equations involving exponents
- Try to get used to 'reading' logarithm statements to yourself
  - $\log_a(b) = x$ would be read as "the power that you raise a to, to get b, is x"
  - So $\log_5 125 = 3$ would be read as "the power that you raise 5 to, to get 125, is 3"
- Two important cases are:
  - $\ln x = \log_e(x)$
    - Where e is the mathematical constant 2.718…
    - This is called the natural logarithm and will have its own button on your GDC
  - $\log x = \log_{10}(x)$
    - Logarithms of base 10 are used often and so abbreviated to log *x*

Why use logarithms?

- Logarithms allow us to solve equations where the exponent is the unknown value
  - We can solve some of these by inspection
    - For example, for the equation $2^x = 8$ we know that *x* must be 3
  - Logarithms allow use to solve more complicated problems
    - For example, the equation $2^x = 10$ does not have a clear answer
    - Instead, we can use our GDCs to find the value of $\log_2 10$

Exam Tip

- Before going into the exam, make sure you are completely familiar with your GDC and know how to use its logarithm functions

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Worked Example

Solve the following equations:

i)
$x = \log_3 27$,

$x = \log_3 27 \iff 3^x = 27$

We can see from inspection:

$3^3 = 27 \iff x = 3$

$\boxed{x = 3}$

OR: use GDC to find answer directly.

ii)
$2^x = 21.4$, giving your answer to 3 s.f.

$2^x = 21.4$ This cannot be seen from inspection:

$2^x = 21.4 \iff x = \log_2 21.4$

Use GDC to find answer directly.

$\log_2 21.4 = 4.4195...$

$\boxed{x = 4.42 \text{ (3 s.f.)}}$

1.2.2 Laws of Logarithms

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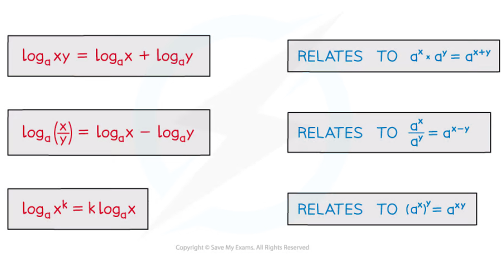
Laws of Logarithms

What are the laws of logarithms?

- Laws of logarithms allow you to simplify and manipulate expressions involving logarithms
  - The laws of logarithms are equivalent to the **laws of indices**
- The laws you need to know are, given $a > 0$:
  - $\log_a xy = \log_a x + \log_a y$
    - This relates to $a^x \times a^y = a^{x+y}$
  - $\log_a \frac{x}{y} = \log_a x - \log_a y$
    - This relates to $a^x \div a^y = a^{x-y}$
  - $\log_a x^m = m\log_a x$
    - This relates to $(a^x)^y = a^{xy}$
- These laws are **in the formula booklet** so you do not need to remember them
  - You must make sure you know how to use them



Useful results from the laws of logarithms

- Given $a > 0$, $a \neq 1$
  - $\log_a 1 = 0$
    - This is equivalent to $a^0 = 1$
- If we substitute *b* for *a* into the given identity in the formula booklet
  - $a^x = b \Leftrightarrow \log_a b = x$ where $a > 0,\ b > 0,\ a \neq 1$
  - $a^x = a \Leftrightarrow \log_a a = x$ gives $a^1 = a \Leftrightarrow \log_a a = 1$
    - This is an important and useful result
- Substituting this into the third law gives the result
  - $\log_a a^k = k$
- Taking the inverse of its operation gives the result
  - $a^{\log_a x} = x$
- From the third law we can also conclude that
  - $\log_a \frac{1}{x} = -\log_a x$

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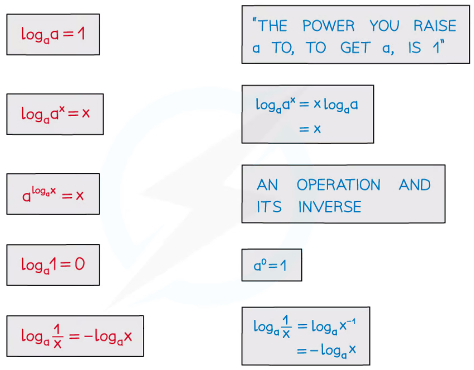

- These useful results are **not in the formula booklet** but can be deduced from the laws that are
- Beware...
  - ...$\log_a(x + y) \neq \log_a x + \log_a y$
- These results apply to $\ln x$ $(\log_e x)$ too
  - Two particularly useful results are
    - $\ln e^x = x$
    - $e^{\ln x} = x$
- Laws of logarithms can be used to ...
  - simplify expressions
  - solve logarithmic equations
  - solve exponential equations

Exam Tip

- Remember to check whether your solutions are valid
  - log (x+k) is only defined if x > -k
  - You will lose marks if you forget to reject invalid solutions

YOUR NOTES

Worked Example

a)
Write the expression $2\log 4 - \log 2$ in the form $\log k$, where $k \in \mathbb{Z}$.

Using the law $\log_a x^m = m\log_a x$

$2\log 4 = \log 4^2 = \log 16$

$2\log 4 - \log 2 = \log 4^2 - \log 2$

$= \log 16 - \log 2$

Using the law $\log_a \frac{x}{y} = \log_a x - \log_a y$

$\log 16 - \log 2 = \log \frac{16}{2} = \log 8$

$$\boxed{2\log 4 - \log 2 = \log 8}$$

b) Hence, or otherwise, solve $2\log 4 - \log 2 = -\log \frac{1}{x}$.

To solve $2\log 4 - \log 2 = \log\frac{1}{x}$ rewrite as

$\log 8 = -\log \frac{1}{x}$

from part (a)

Use the index law $\frac{1}{x} = x^{-1}$

$\log 8 = -\log x^{-1}$

$\log 8 = \log x$ $\log_a x^m = m\log_a x$

$8 = x$

$$\boxed{x = 8}$$

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Change of Base

Why change the base of a logarithm?

- The laws of logarithms can only be used if the logs have the same **base**
  - If a problem involves logarithms with different bases, you can change the base of the logarithm and then apply the laws of logarithms
- **Changing the base** of a logarithm can be particularly useful if you need to evaluate a log problem **without a calculator**
  - Choose the base such that you would know how to solve the problem from the equivalent exponent

How do I change the base of a logarithm?

- The formula for changing the base of a logarithm is

$$\log_a x = \frac{\log_b x}{\log_b a}$$

- This is **in the formula booklet**
- The value you choose for *b* does not matter, however if you do not have a calculator, you can choose *b* such that the problem will be possible to solve

Exam Tip

- Changing the base is a key skill which can help you with many different types of questions, make sure you are confident with it
  - It is a particularly useful skill for examinations where a GDC is not permitted

YOUR NOTES

Worked Example

By choosing a suitable value for *b*, use the change of base law to find the value of $\log_8 32$ without using a calculator.

Change of base law: $\log_a x = \dfrac{\log_b x}{\log_b a}$

$\log_8 32$

$2^5 = 32$

$2^3 = 8$

Choose $b = 2$ to allow for a solution by inspection

$$\log_8 32 = \frac{\log_2 32}{\log_2 8} = \frac{5}{3}$$

$$\boxed{\log_8 32 = 1\tfrac{2}{3}}$$

YOUR NOTES

Solving Exponential Equations

What are exponential equations?

- An exponential equation is an equation where the unknown is a power
  - In simple cases the solution can be spotted without the use of a calculator
  - For example,

$$
\begin{aligned}
5^{2x} &= 125 \\
2x &= 3 \\
x &= \frac{3}{2}
\end{aligned}
$$

- In more complicated cases the laws of logarithms should be used to solve exponential equations

- The **change of base** law can be used to solve some exponential equations without a calculator
  - For example,

$$
\begin{aligned}
27^{x} &= 9 \\
x &= \log_{27} 9 \\
&= \frac{\log_3 9}{\log_3 27} \\
&= \frac{2}{3}
\end{aligned}
$$

How do we use logarithms to solve exponential equations?

- An exponential equation can be solved by taking logarithms of both sides
- The **laws of indices** may be needed to rewrite the equation first
- The **laws of logarithms** can then be used to solve the equation
  - **ln ($\log_e$)** is often used
  - The answer is often written in terms of ln
- A question my ask you to give your answer in a particular form
- Follow these steps to solve exponential equations
  - STEP 1: Take logarithms of both sides
  - STEP 2: Use the laws of logarithms to remove the powers
  - STEP 3: Rearrange to isolate *x*
  - STEP 4: Use logarithms to solve for *x*

What about hidden quadratics?

- Look for hidden squared terms that could be changed to form a quadratic
  - In particular look out for terms such as
    - $4^x = (2^2)^x = 2^{2x} = (2^x)^2$
    - $e^{2x} = (e^2)^x = (e^x)^2$

YOUR NOTES
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Exam Tip

- Always check which form the question asks you to give your answer in, this can help you decide how to solve it
- If the question requires an exact value you may need to leave your answer as a logarithm

Worked Example

Solve the equation $4^x - 3(2^{x+1}) + 9 = 0$. Give your answer correct to three significant figures.

Spot the hidden quadratic: $4^x = (2^2)^x = (2^x)^2$

By the laws of indices $2^{x+1} = 2^x \times 2^1 = 2 \times 2^x$

$(2^x)^2 - 3(2^{x+1}) + 9 = 0$

$(2^x)^2 - 3 \times 2 \times 2^x + 9 = 0$

$(2^x)^2 - 6 \times 2^x + 9 = 0$

Let $u = 2^x$ $\quad u^2 - 6u + 9 = 0$

$(u-3)(u-3) = 0$

$u = 3 \quad \therefore \quad 2^x = 3$

Solve the exponential equation $2^x = 3$

Step 1: Take logarithms of both sides: $\ln(2^x) = \ln(3)$

Step 2: Use the law $\log_a x^m = m\log_a x$ $\quad x\ln 2 = \ln 3$

Step 3: Rearrange to isolate x $\quad x = \frac{\ln 3}{\ln 2}$

Step 4: Solve $\quad x = \frac{\ln 3}{\ln 2} = 1.584\ldots$

$\boxed{x = 1.58 \text{ (3s.f.)}}$

YOUR NOTES
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1.3 Sequences & Series

1.3.1 Language of Sequences & Series

Language of Sequences & Series

What is a sequence?

- A **sequence** is an ordered set of numbers with a rule for finding all of the numbers in the sequence
  - For example 1, 3, 5, 7, 9, … is a sequence with the rule *'start at one and add two to each number'*
- The numbers in a sequence are often called **terms**
- The terms of a sequence are often referred to by letters with a subscript
  - In IB this will be the letter u
  - So in the sequence above, $u_1 = 1$, $u_2 = 3$, $u_3 = 5$ and so on
- Each term in a sequence can be found by **substituting** the term number into **formula for the nth term**

What is a series?

- You get a **series** by summing up the terms in a sequence
  - E.g. For the sequence 1, 3, 5, 7, … the associated series is $1 + 3 + 5 + 7 + \ldots$
- We use the notation S_n to refer to the sum of the first n terms in the series
  - $S_n = u_1 + u_2 + u_3 + \ldots + u_n$
  - So for the series above $S_5 = 1 + 3 + 5 + 7 + 9 = 25$

What are increasing, decreasing and periodic sequences?

- A sequence is **increasing** if $u_{n+1} > u_n$ for all positive integers
  - i.e. every term is greater than the term before it
- A sequence is **decreasing** if $u_{n+1} < u_n$ for all positive integers
  - i.e. every term is less than the term before it
- A sequence is **periodic** if the terms repeat in a cycle

YOUR NOTES

Worked Example

Determine the first five terms and the value of S_5 in the sequence with terms defined by $u_n = 5 - 2n$.

$u_n = 5 - 2n$

find the term you want by replacing n with it's value.

term number

first term

$u_1 = 5 - 2(1) = 3$

$u_2 = 5 - 2(2) = 1$

$u_3 = 5 - 2(3) = -1$

$u_4 = 5 - 2(4) = -3$

$u_5 = 5 - 2(5) = -5$

recognise the pattern.

-2

-2

rule is subtract 2

'start with 3 and subtract 2 from each number'.

$S_5 = 3 + 1 + (-1) + (-3) + (-5) = -5$

the sum of the first 5 terms

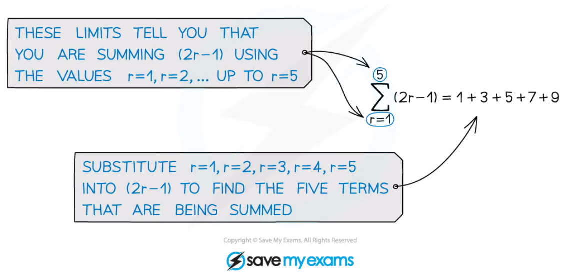
$$\boxed{\begin{array}{c} 3,\ 1,\ -1,\ -3,\ -5 \\ S_5 = -5 \end{array}}$$

Sigma Notation

YOUR NOTES

What is sigma notation?

- Sigma notation is used to show the sum of a certain number of terms in a sequence
- The symbol Σ is the capital Greek letter sigma
- Σ stands for 'sum'
    - The expression to the right of the Σ tells you what is being summed, and the limits above and below tell you which terms you are summing



- Be careful, the limits don't have to start with 1
    - For example $\sum_{k=0}^{4}(2k+1)$ or $\sum_{k=7}^{14}(2k-13)$
    - *r* and *k* are commonly used variables within sigma notation

Exam Tip

- Your GDC will be able to use sigma notation, familiarise yourself with it and practice using it to check your work

YOUR NOTES

Worked Example

A sequence can be defined by $u_n = 2 \times 3^{n-1}$ for $n \in \mathbb{Z}^+$.

a)
Write an expression for $u_1 + u_2 + u_3 + \ldots + u_6$ using sigma notation.

$u_n = 2 \times 3^{n-1}, \; n \in \mathbb{Z}^+$ ← n is the set of all positive integers

Using sigma notation

$$u_1 + u_2 + \ldots + u_6 = \sum_{k=1}^{6} u_k$$

$$\boxed{\sum_{k=1}^{6} (2 \times 3^{k-1})}$$

b)
Write an expression for $u_7 + u_8 + u_9 + \ldots + u_{12}$ using sigma notation.

$u_n = 2 \times 3^{n-1}, \; n \in \mathbb{Z}^+$ ← n is the set of all positive integers

Using sigma notation

$$u_7 + u_8 + \ldots + u_{12} = \sum_{k=7}^{12} u_k$$

$$\boxed{\sum_{k=7}^{12} (2 \times 3^{k-1})}$$

1.3.2 Arithmetic Sequences & Series

YOUR NOTES

Arithmetic Sequences

What is an arithmetic sequence?

- In an **arithmetic sequence,** the difference between consecutive terms in the sequence is constant
- This **constant difference** is known as the **common difference, *d*,** of the sequence
  - For example, 1, 4, 7, 10, … is an arithmetic sequence with the rule '*start at one and add three to each number*'
    - The **first term, u_1,** is 1
    - The **common difference, *d*,** is 3
  - An arithmetic sequence can be **increasing** (positive common difference) or **decreasing** (negative common difference)
  - Each term of an arithmetic sequence is referred to by the letter *u* with a subscript determining its place in the sequence

How do I find a term in an arithmetic sequence?

- The n^{th} term formula for an arithmetic sequence is given as

$$u_n = u_1 + (n-1)d$$

  - Where u_1 is the first term, and d is the common difference
  - This is given in the formula booklet, you do not need to know how to derive it
- Enter the information you have into the formula and use your GDC to find the value of the term
- Sometimes you will be given a term and asked to find the first term or the common difference
  - Substitute the information into the formula and solve the equation
    - You could use your GDC for this
- Sometimes you will be given two terms and asked to find both the first term and the common difference
  - Substitute the information into the formula and set up a **system of linear equations**
  - Solve the simultaneous equations
    - You could use your GDC for this

Exam Tip

- Simultaneous equations are often needed within arithmetic sequence questions, make sure you are confident solving them with and without the GDC

YOUR NOTES

Worked Example

The fourth term of an arithmetic sequence is 10 and the ninth term is 25, find the first term and the common difference of the sequence.

$u_4 = 10 \quad , \quad u_9 = 25$

Formula for n^{th} term of an arithmetic series:

$u_n = u_1 + (n-1)d$

Sub in $u_4 = 10$ and $u_9 = 25$

$u_4 = u_1 + (4-1)d = u_1 + 3d = 10$

$u_9 = u_1 + (9-1)d = u_1 + 8d = 25$

Solve using GDC:

let $u_1 = x$ and $d = y$

$x + 3y = 10$
$x + 8y = 25$

$x = 1 \quad , \quad y = 3$

$$\boxed{\begin{aligned} u_1 &= 1 \\ d &= 3 \end{aligned}}$$

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Arithmetic Series

How do I find the sum of an arithmetic series?

- An **arithmetic series** is the sum of the terms in an **arithmetic sequence**
  - For the arithmetic sequence 1, 4, 7, 10, ... the arithmetic series is 1 + 4 + 7 + 10 + ...
- Use the following formulae to find the sum of the first n terms of the arithmetic series:

$$S_n = \frac{n}{2}(2u_1 + (n-1)d) \quad ; \quad S_n = \frac{n}{2}(u_1 + u_n)$$

    - u_1 is the first term
    - d is the common difference
    - u_n is the last term
  - Both formulae are given in the formula booklet, you do not need to know how to derive them
- You can use whichever formula is more convenient for a given question
  - If you know the first term and common difference use the first version
  - If you know the first and last term then the second version is easier to use
- A question will often give you the sum of a certain number of terms and ask you to find the value of the first term or the common difference
  - Substitute the information into the formula and solve the equation
    - You could use your GDC for this

Exam Tip

- The formulae you need for arithmetic series are in the formula book, you do not need to remember them
  - Practice finding the formulae so that you can quickly locate them in the exam

YOUR NOTES
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Worked Example

The sum of the first 10 terms of an arithmetic sequence is 630.

a)
Find the common difference, *d*, of the sequence if the first term is 18.

$S_{10} = 630$

Formula for the sum of an arithmetic series:

$$S_n = \frac{n}{2}(2u_1 + (n-1)d)$$

Sub in $S_{10} = 630$, $u_1 = 18$

$$S_{10} = \frac{10}{2}(2(18) + (10-1)d) = 630$$

$$5(36 + 9d) = 630$$

Solve: $36 + 9d = 126$

$$9d = 90$$

$$d = 10$$

$$\boxed{d = 10}$$

b)
Find the first term of the sequence if the common difference, *d*, is 11.

Sub in $S_{10} = 630$, $d = 11$

$$S_{10} = \frac{10}{2}(2u_1 + (10-1)(11)) = 630$$

$$5(2u_1 + 99) = 630$$

Solve: $2u_1 + 99 = 126$

$$2u_1 = 27$$

$$\boxed{u_1 = 13.5}$$

YOUR NOTES
↓

1.3.3 Geometric Sequences & Series

YOUR NOTES

Geometric Sequences

What is a geometric sequence?

- In a **geometric sequence**, there is a **common ratio, r,** between consecutive terms in the sequence
  - For example, 2, 6, 18, 54, 162, ... is a sequence with the rule '*start at two and multiply each number by three*'
    - The **first term, u_1,** is 2
    - The **common ratio, r,** is 3
- A geometric sequence can be **increasing** ($r > 1$) or **decreasing** ($0 < r < 1$)
- If the common ratio is a **negative number** the terms will alternate between positive and negative values
  - For example, 1, -4, 16, -64, 256, ... is a sequence with the rule '*start at one and multiply each number by negative four*'
    - The **first term, u_1,** is 1
    - The **common ratio, r,** is -4
- Each term of a geometric sequence is referred to by the letter u with a subscript determining its place in the sequence

How do I find a term in a geometric sequence?

- The n^{th} term formula for a geometric sequence is given as

$$u_n = u_1 r^{n-1}$$

  - Where u_1 is the first term, and r is the common ratio
  - This formula allows you to find **any term** in the geometric sequence
  - It is given in the formula booklet, you do not need to know how to derive it
- Enter the information you have into the formula and use your GDC to find the value of the term
- Sometimes you will be given a term and asked to find the first term or the common ratio
  - Substitute the information into the formula and solve the equation
    - You could use your GDC for this
- Sometimes you will be given two or more consecutive terms and asked to find both the first term and the common ratio
  - Find the common ratio by dividing a term by the one before it
  - Substitute this and one of the terms into the formula to find the first term
- Sometimes you may be given a term and the formula for the n^{th} term and asked to find the value of n
  - You can solve these using **logarithms** on your GDC

YOUR NOTES

Exam Tip

- You will sometimes need to use logarithms to answer geometric sequences questions
  - Make sure you are confident doing this
  - Practice using your GDC for different types of questions

YOUR NOTES

Worked Example

The sixth term, u_6, of a geometric sequence is 486 and the seventh term, u_7, is 1458.

Find,

i)
the common ratio, r, of the sequence,

$u_6 = 486 \quad , \quad u_7 = 1458$

The common ratio, r, is given by

$$r = \frac{u_2}{u_1} = \frac{u_3}{u_2} = \ldots = \frac{u_{n+1}}{u_n}$$

Sub in $u_6 = 486$, $u_7 = 1458$

$$r = \frac{u_7}{u_6} = \frac{1458}{486} = 3$$

$$\boxed{r = 3}$$

ii)
the first term of the sequence, u_1.

Formula for n^{th} term of a geometric series:

$$u_n = u_1 r^{n-1}$$

Sub in $r = 3$ and either $u_6 = 486$ or $u_7 = 1458$

$$u_6 = u_1(3)^{6-1} = 486$$

Solve: $243\, u_1 = 486$

$u_1 = 2$

$$\boxed{u_1 = 2}$$

YOUR NOTES
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Geometric Series

How do I find the sum of a geometric series?

- A **geometric series** is the sum of a certain number of terms in a **geometric sequence**
  - For the geometric sequence 2, 6, 18, 54, ... the geometric series is 2 + 6 + 18 + 54 + ...
- The following formulae will let you find the sum of the first n terms of a geometric series:

$$S_n = \frac{u_1(r^n - 1)}{r - 1} = \frac{u_1(1 - r^n)}{1 - r}$$

    - u_1 is the first term
    - r is the common ratio
  - Both formulae are given in the formula booklet, you do not need to know how to derive them
- You can use whichever formula is more convenient for a given question
  - The first version of the formula is more convenient if $r > 1$ and the second is more convenient if $r < 1$
- A question will often give you the sum of a certain number of terms and ask you to find the value of the first term, the common ratio, or the number of terms within the sequence
  - Substitute the information into the formula and solve the equation
    - You could use your GDC for this

Exam Tip

- The geometric series formulae are in the formulae booklet, you don't need to memorise them
  - Make sure you can locate them quickly in the formula booklet

YOUR NOTES

Worked Example

A geometric sequence has $u_1 = 25$ and $r = 0.8$. Find the value of u_5 and S_5.

$u_1 = 25, \quad r = 0.8$

Formula for n^{th} term of a geometric series:

$$u_n = u_1 r^{n-1}$$

Sub in $u_1 = 25, \quad r = 0.8$

$$u_5 = 25(0.8)^4 = 10.24$$

Formula for the sum of a geometric series:

$$S_n = \frac{u_1(r^n - 1)}{r - 1} = \frac{u_1(1 - r^n)}{1 - r}$$

$r < 1$ so this version is easier to use.

Sub in $u_1 = 25, \quad r = 0.8$

$$S_5 = \frac{u_1(1 - r^5)}{1 - r} = \frac{25(1 - 0.8^5)}{1 - 0.8} = 84.04$$

$$\boxed{\begin{aligned} u_5 &= 10.24 \\ S_5 &= 84.04 \end{aligned}}$$

YOUR NOTES
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Sum to Infinity

What is the sum to infinity of a geometric series?

- A geometric sequence will either increase or decrease away from zero or the terms will get progressively closer to zero
  - Terms will get closer to zero if the common ratio, *r*, is between 1 and -1
- If the terms are getting closer to zero then the series is said to **converge**
  - This means that the sum of the series will approach a limiting value
  - As the number of terms increase, the sum of the terms will get closer to the limiting value

How do we calculate the sum to infinity?

- If asked to find out if a geometric sequence converges find the value of *r*
  - If $|r| < 1$ then the sequence converges
  - If $|r| \geq 1$ then the sequence does not converge and the sum to infinity cannot be calculated
  - $|r| < 1$ means $-1 < r < 1$
- If $|r| < 1$, then the geometric series **converges** to a finite value given by the formula

$$S_{\infty} = \frac{u_1}{1 - r}, \quad |r| < 1$$

  - u_1 is the first term
  - r is the common ratio
  - This is **in the formula book**, you do not need to remember it

Exam Tip

- Learn and remember the conditions for when a sum to infinity can be calculated

YOUR NOTES
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Worked Example

The first three terms of a geometric sequence are $6\,,\ 2\,,\ \frac{2}{3}$. Explain why the series converges and find the sum to infinity.

$u_1 = 6, \quad u_2 = 2, \quad u_3 = \frac{2}{3}$

Find the value of r: $r = \frac{u_2}{u_1}$

$$r = \frac{u_2}{u_1} = \frac{2}{6} = \frac{1}{3}$$

$\boxed{|r| < 1 \text{ so the series converges}}$

Find the sum to infinity: $S_\infty = \frac{u_1}{1-r}$

$$S_\infty = \frac{u_1}{1-r} = \frac{6}{1-\frac{1}{3}} = \frac{6}{2/3} = 9$$

$$\boxed{S_\infty = 9}$$

1.3.4 Applications of Sequences & Series

YOUR NOTES

Applications of Arithmetic Sequences & Series

Many real-life situations can be modelled using sequences and series, including but not limited to: patterns made when tiling floors; seating people around a table; the rate of change of a population; the spread of a virus and many more.

What do I need to know about applications of arithmetic sequences and series?

- If a quantity is changing repeatedly by having a fixed amount **added to** or **subtracted from** it then the use of **arithmetic sequences** and **arithmetic series** is appropriate to **model** the situation
    - If a sequence seems to fit the pattern of an arithmetic sequence it can be said to be **modelled** by an arithmetic sequence
    - The scenario can be **modelled** using the given information and the formulae from the formula booklet
- A common application of arithmetic sequences and series is **simple interest**
    - Simple interest is when an initial investment is made and then a percentage of the initial investment is added to this amount on a regular basis (usually per year)
- Arithmetic sequences can be used to make estimations about how something will change in the future

Exam Tip

- Exam questions won't always tell you to use sequences and series methods, practice spotting them by looking for clues in the question
- If a given amount is repeated periodically then it is likely the question is on arithmetic sequences or series

YOUR NOTES

Worked Example

Jasper is saving for a new car. He puts USD $100 into his savings account and then each month he puts in USD $10 more than the month before. Jasper needs USD $1200 for the car. Assuming no interest is added, find,

i)
the amount Jasper has saved after four months,

Identify the arithmetic sequence:

$u_1 = 100 \quad , \quad d = 10$

After 4 months Jasper will have saved:

$u_1 + u_2 + u_3 + u_4 = S_4$

Formula for the sum of an arithmetic series:

$S_n = \frac{n}{2}(2u_1 + (n-1)d)$

$S_4 = \frac{4}{2}(2u_1 + (4-1)d)$

Sub in $u_1 = 100$ and $d = 10$

$S_4 = \frac{4}{2}(2(100) + (4-1)(10))$

$= 2(200 + 30)$

$= 2(230)$

$\boxed{S_4 = \$460}$

ii)
the month in which Jasper reaches his goal of USD $1200.

YOUR NOTES

Sub $S_n = 1200$, $u_1 = 100$, $d = 10$ into formula:

$$1200 = \frac{n}{2}(2(100) + (n-1)(10))$$

Solve using algebraic solver on GDC:

$n = 8.67...$ or $n = -27.67...$

disregard as n cannot be negative.

$\therefore$ $S_8 < 1200$

$S_9 > 1200$ reaches total in 9^{th} month

Jasper will reach USD \$1200 in the 9^{th} month.

Applications of Geometric Sequences & Series

YOUR NOTES
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What do I need to know about applications of geometric sequences and series?

- If a quantity is changing repeatedly by a fixed **percentage,** or by being **multiplied** repeatedly by a fixed amount, then the use of **geometric sequences** and **geometric series** is appropriate to **model** the situation
  - If a sequence seems to fit the pattern of a geometric sequence it can be said to be **modelled** by a geometric sequence
  - The scenario can be **modelled** using the given information and the formulae from the formula booklet
- A common application of geometric sequences and series is **compound interest**
  - Compound interest is when an initial investment is made and then interest is paid on the initial amount **and on the interest already earned** on a regular basis (usually every year)
- Geometric sequences can be used to make estimations about how something will change in the future
- The questions won't always tell you to use sequences and series methods, so be prepared to spot 'hidden' sequences and series questions
  - Look out for questions on savings accounts, salaries, sales commissions, profits, population growth and decay, spread of bacteria etc

Exam Tip

- Exam questions won't always tell you to use sequences and series methods, practice spotting them by looking for clues in the question
- If a given amount is changing by a percentage or multiple then it is likely the question is on geometric sequences or series

YOUR NOTES

Worked Example

A new virus is circulating on a remote island. On day one there were 10 people infected, with the number of new infections increasing at a rate of 40% per day.

a)

Find the expected number of people newly infected on the 7th day.

Identify the geometric sequence:

$u_1 = 10$, $r = 1.4$ ← 40% increase so 140% of the day before

New infections : u_7

Formula for n^{th} term of a geometric series :

$$u_n = u_1 r^{n-1}$$

Sub in $u_1 = 10$, $r = 1.4$

$$u_7 = 10(1.4)^6 = 75.29\ldots$$

$$\boxed{\text{Expected number of new infections } = 75}$$

b)

Find the expected number of infected people after one week (7 days), assuming no one has recovered yet.

Total infections: S_7

Formula for the sum of a geometric series:

$$S_n = \frac{u_1(r^n - 1)}{r - 1}$$ ← $r > 1$ so this version is easier to use.

Sub in $u_1 = 10$, $r = 1.4$

$$S_7 = \frac{10(1.4^7 - 1)}{1.4 - 1} = 238.53\ldots$$

$$\boxed{\text{Expected number of total infections } = 239}$$

1.3.5 Compound Interest & Depreciation

YOUR NOTES

Compound Interest

What is compound interest?

- Interest is a small percentage paid by a bank or company that is added on to an initial investment
  - Interest can also refer to an amount paid on a loan or debt, however IB compound interest questions will always refer to interest on **investments**
- **Compound interest** is where interest is paid on **both the initial investment** and any interest that has **already been paid**
  - Make sure you know the difference between compound interest and simple interest
    - Simple interest pays interest only on the initial investment
- The interest paid each time will increase as it is a percentage of a higher number
- Compound interest will be paid in instalments in a given timeframe
  - The interest rate, *r*, will be per annum (per year)
    - This could be written *r* % p.a.
  - Look out for phrases such as **compounding annually** (interest paid yearly) **or compounding monthly** (interest paid monthly)
    - If $\alpha\%$ p.a. (per annum) is paid compounding monthly, then $\frac{\alpha}{12}\%$ will be paid each month
    - The formula for compound interest allows for this so you do not have to compensate separately

How is compound interest calculated?

- The formula for calculating compound interest is:

$$FV = PV \times \left(1 + \frac{r}{100k}\right)^{kn}$$

  - Where
    - *FV* is the future value
    - *PV* is the present value
    - *n* is the number of years
    - *k* is the number of compounding periods per year
    - *r* % is the nominal annual rate of interest
  - This formula is **given in the formula booklet**, you do not have to remember it
- Be careful with the *k* value
  - Compounding annually means *k* = 1
  - Compounding half-yearly means *k* = 2
  - Compounding quarterly means *k* = 4
  - Compounding monthly means *k* = 12
- Your GDC will have a finance solver app on it which you can use to find the future value
  - This may also be called the TVM (time value of money) solver
  - You will have to enter the information from the question into your calculator
- Be aware that many questions will be set up such that you will have to use the formula

- So for compound interest questions it is better to use the formula from your formula booklet than your GDC

YOUR NOTES

Exam Tip

- Your GDC will be able to solve some compound interest problems so it is a good idea to make sure you are confident using it, however you must also familiarise yourself with the formula and make sure you can find it in the formula booklet

Worked Example

Kim invests MYR 2000 (Malaysian Ringgit) in an account that pays a nominal annual interest rate of 2.5% **compounded monthly**. Calculate the amount that Kim will have in her account after 5 years.

Compound interest formula:

$$FV = PV\left(1 + \frac{r}{100k}\right)^{kn}$$

future value, present value, interest rate, compounding periods, number of years

Substitute values in:

$PV = 2000$ (initial investment)

$k = 12$ (compounding monthly)

$r = 2.5\%$

$n = 5$ (number of years)

$$FV = 2000\left(1 + \frac{2.5}{(100)(12)}\right)^{(12 \times 5)}$$

$$= 2266.002\ldots$$

$$\boxed{FV \approx \text{MYR } 2270 \text{ (3sf)}}$$

Depreciation

YOUR NOTES

What is depreciation?

- Depreciation is when something looses value over time
    - The most common examples of depreciation are the value of cars and technology or the temperature of a cooling cup of coffee

How is compound depreciation calculated?

- The formula for calculating compound depreciation is:

$$FV = PV \times \left(1 - \frac{r}{100}\right)^n$$

- Where
    - FV is the future value
    - PV is the present value
    - n is the number of years
    - r% is the rate of depreciation
  - This formula is **not** given in the formula booklet, however it is almost the same as the formula for compound interest but
    - with a **subtraction** instead of an addition
    - the value of k will always be 1
- Your GDC **could** again be used to solve some compound depreciation questions, but watch out for those which are set up such that you will have to use the formula

Exam Tip

- Although the formula is not the same as the one given in the formula booklet for compound interest, it is very similar
    - Practice finding this formula and recognising the differences

YOUR NOTES

Worked Example

Kyle buys a new car for AUD $14 999. The value of the car depreciates by 15% each year.

a)

Find the value of the car after 5 years.

Depreciation formula:

$$FV = PV\left(1 - \frac{r}{100}\right)^n$$

rate of depreciation (r)

n ← number of years

future value (FV)

present value (PV)

Substitute values in:

$PV = 14999$ (initial cost)

$r = 15\%$

$n = 5$ (number of years)

$$FV = 14999\left(1 - \frac{15}{100}\right)^5$$

$$= 6655.13\ldots$$

$$\boxed{FV \approx \text{AUD } \$6660 \text{ (3sf)}}$$

b)

Find the number of years and months it will take for the value of the car to be approximately AUD $9999.

$$FV = PV\left(1 - \frac{r}{100}\right)^n$$

$FV \approx 9999$

$PV = 14999$

$r = 15\%$

Substitute values in:

$$9999 \approx 14999\left(1 - \frac{15}{100}\right)^n$$

Use GDC to solve:

$n = 2.495...$

2 years 0.495^{th} of a year

Convert to years and months:

2 years + 0.495... × 12 months

$$\boxed{\approx 2 \text{ years and } 6 \text{ months}}$$

YOUR NOTES

YOUR NOTES
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1.4 Proof & Reasoning

1.4.1 Proof

Language of Proof

What is proof?

- Proof is a series of logical steps which show a result is **true** for all specified numbers
  - 'Seeing' that a result works for a few numbers is not enough to **show** that it will work for all numbers
  - Proof allows us to show (usually algebraically) that the result will work for **all values**
- You must be familiar with the notation and language of proof
- LHS and RHS are standard abbreviations for left-hand side and right-hand side
- **Integers** are used frequently in the language of proof
  - The set of **integers** is denoted by $\mathbb{Z}$
  - The set of **positive integers** is denoted by $\mathbb{Z}^+$

How do we prove a statement is true for all values?

- Most of the time you will need to use algebra to show that the left-hand side (LHS) is the same as the right-hand side (RHS)
  - You **must not** move terms from one side to the other
  - Start with one side (usually the LHS) and manipulate it to show that it is the same as the other
- A **mathematical identity** is a statement that is true for all values of x (or θ in trigonometry)
  - The symbol $\equiv$ is used to identify an identity
  - If you see this symbol then you can use proof methods to show it is true
- You can complete your proof by stating that RHS = LHS or writing QED

Exam Tip

- You will need to show each step of your proof clearly and set out your method in a logical manner in the exam
  - Be careful not to skip steps

YOUR NOTES

Worked Example

Prove that $(2x-2)(x-3)+2(x-1)=2(x-2)(x-1)$.

Work with LHS first:

Expand brackets:

FOIL

LHS: $(2x-2)(x-3)+2(x-1)$

$2x^2-6x-2x+6+2x-2$

Simplify, take care with negatives:

$2x^2-6x+4$

Factorise the 2:

$2(x^2-3x+2)$

Factorise remaining quadratic:

$2(x-2)(x-1)$ = RHS as required.

$$\boxed{(2x-2)(x-3)+2(x-1)=2(x-2)(x-1)}$$

Proof by Deduction

YOUR NOTES
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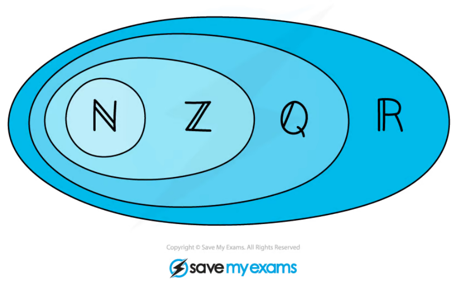
What is proof by deduction?

- A mathematical and logical argument that shows that a result is true

How do we do proof by deduction?

- A proof by deduction question will often involve showing that a result is true for all integers, consecutive integers or even or odd numbers
  - You can begin by letting an integer be n
    - Use conventions for even ($2n$) and odd ($2n - 1$) numbers
- You will need to be familiar with sets of numbers $(\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R})$
  - $\mathbb{N}$ – the set of **natural numbers**
  - $\mathbb{Z}$ – the set of **integers**
  - $\mathbb{Q}$ – the set of **quotients (rational numbers)**
  - $\mathbb{R}$ – the set of **real numbers**



Exam Tip

- Try the result you are proving with a few different values
  - Use a sequence of them (eg 1, 2, 3)
  - Try different types of numbers (positive, negative, zero)
- This may help you see a pattern and spot what is going on

YOUR NOTES

Worked Example

Prove that the sum of any two consecutive odd numbers is always even.

Let $2n - 1$ be an odd number

must be even

Let two consecutive odd integers be:

$2n - 1$, $2n + 1$ next odd number

Then their sum is :

$2n - 1 + 2n + 1 \equiv 4n$

$= 2(2n)$

$\boxed{\text{Any multiple of 2 must be even.}}$

YOUR NOTES
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1.5 Binomial Theorem

1.5.1 Binomial Theorem

Binomial Theorem

What is the Binomial Theorem?

- The **binomial theorem (**sometimes known as the binomial expansion) gives a method for expanding a **two-term** expression in a bracket raised to a power
  - A **binomial expression** is in fact any two terms inside the bracket, however in IB the expression will usually be linear
- To expand a bracket with a two-term expression in:
  - First choose the most appropriate parts of the expression to assign to *a* and *b*
  - Then use the formula for the binomial theorem:

$$(a+b)^n = a^n + {}^nC_1\, a^{n-1} b + \ldots + {}^nC_r\, a^{n-r} b^r + \ldots + b^n$$

  - where ${}^nC_r = \dfrac{n!}{r!(n-r)!}$
    - See below for more information on ${}^nC_r$
    - You may also see ${}^nC_r$ written as $\binom{n}{r}$ or ${}_nC_r$

- You will usually be asked to find the first three or four terms of an expansion
- Look out for whether you should give your answer in **ascending** or **descending** powers of *x*
  - For **ascending** powers start with the constant term, a^n
  - For **descending** powers start with the term with *x* in
    - You may wish to swap *a* and *b* over so that you can follow the general formula given in the formula book
- If you are not writing the full expansion you can either
  - show that the sequence continues by putting an ellipsis (...) after your final term
  - or show that the terms you have found are an approximation of the full sequence by using the sign for approximately equals to (≈)

How do I find the coefficient of a single term?

- Most of the time you will be asked to find the coefficient of a term, rather than carry out the whole expansion
- Use the formula for the general term

$${}^nC_r\, a^{n-r} b^r$$

- The question will give you the power of *x* of the term you are looking for
  - Use this to choose which value of *r* you will need to use in the formula
  - This will depend on where the *x* is in the bracket
  - The laws of indices can help you decide which value of *r* to use:
    - For $(a + bx)^n$ to find the coefficient of x^r use $a^{n-r}(bx)^r$

YOUR NOTES
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- For $(a + bx^2)^n$ to find the coefficient of x^r use $a^{\frac{n-r}{2}}(bx^2)^{\frac{r}{2}}$
- For $(a + \frac{b}{x})^n$ look at how the powers will cancel out to decide which value of r to use
- So for $\left(3x + \frac{2}{x}\right)^8$ to find the coefficient of x^2 use the term with $r = 3$ and to find the constant term use the term with $r = 4$
- There are a lot of variations of this so it is usually easier to see this by inspection of the exponents

- You may also be given the coefficient of a particular term and asked to find an unknown in the brackets
  - Use the laws of indices to choose the correct term and then use the binomial theorem formula to form and solve and equation

Exam Tip

- Binomial expansion questions can get messy, use separate lines to keep your working clear and always put terms in brackets

Worked Example

Find the first three terms, in ascending powers of x, in the expansion of $(3 - 2x)^5$.

$a = 3 \quad b = -2x \quad n = 5$

Substitute values into the formula for $(a+b)^n$

$(a+b)^n = a^n + {}^nC_1 a^{n-1} b + \ldots + {}^nC_r a^{n-r} b^r + \ldots + b^n$

Question asks for ascending powers of x, so start with the constant term, a^n.

$(3 - 2x)^5 = 3^5 + {}^5C_1 (3)^{5-1}(-2x) + {}^5C_2 (3)^{5-2}(-2x)^2 + \ldots$

Watch out for the negative

$\approx 243 + 5 \times 81 \times -2x + 10 \times 27 \times 4x^2$

$\approx 243 - 810x + 1080x^2$

$\boxed{(3 - 2x)^5 \approx 243 - 810x + 1080x^2}$

The Binomial Coefficient nCr

YOUR NOTES
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What is $^{n}C_{r}$?

- If we want to find the number of ways to **choose** *r* items out of *n* different objects we can use the formula for $^{n}C_{r}$
  - The formula for *r* **combinations** of *n* items is $^{n}C_{r} = \frac{n!}{r!(n-r)!}$
  - This formula is given in the formula booklet along with the formula for the binomial theorem
  - The function $^{n}C_{r}$ can be written $\binom{n}{r}$ or $_{n}C_{r}$ and is often read as '*n choose r*'
    - Make sure you can find and use the button on your GDC

How does $^{n}C_{r}$ relate to the binomial theorem?

- The formula $^{n}C_{r} = \frac{n!}{r!(n-r)!}$ is also known as a **binomial** coefficient
- For a binomial expansion $(a + b)^{n}$ the coefficients of each term will be $^{n}C_{0}$, $^{n}C_{1}$ and so on up to $^{n}C_{n}$
  - The coefficient of the r^{th} term will be $^{n}C_{r}$
- $^{n}C_{n} = {}^{n}C_{0} = 1$
- The binomial coefficients are symmetrical, so $^{n}C_{r} = {}^{n}C_{n-r}$
  - This can be seen by considering the formula for $^{n}C_{r}$
  - $^{n}C_{n-r} = \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{r!(n-r)!} = nC_{r}$

Exam Tip

- You will most likely need to use the formula for nCr at some point in your exam
  - Practice using it and don't always rely on your GDC
  - Make sure you can find it easily in the formula booklet

YOUR NOTES
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Worked Example

Without using a calculator, find the coefficient of the term in x^3 in the expansion of $(1+x)^9$.

$n=9, \quad a=1, \quad b=x$

Substitute values into the formula for the binomial theorem:

$(a+b)^n = a^n + \ldots + {}^nC_r\, a^{n-r} b^r + \ldots + b^n$

where ${}^nC_r = \dfrac{n!}{r!(n-r)!}$

$(1+x)^9 = \sum_{r=0}^{9} {}^9C_r (1)^{9-r} (x)^r$ ← Coefficient of x^3 occurs when $r=3$.

$r=3$ gives ${}^9C_3 \times (1)^{9-3} (x)^3$

Non-calculator, so work out ${}^nC_r$ separately:

$${}^9C_3 = \frac{9!}{3!(9-3)!} = \frac{9\times8\times7\times\cancel{6}\times\cancel{5}\times\cancel{4}\times\cancel{3}\times\cancel{2}}{(3\times2)(\cancel{6}\times\cancel{5}\times\cancel{4}\times\cancel{3}\times\cancel{2})}$$

$$= \frac{9\times8\times7}{6} = 84$$

so the term when $r=3$ is $84\times(1)^6\times x^3$

$= 84x^3$

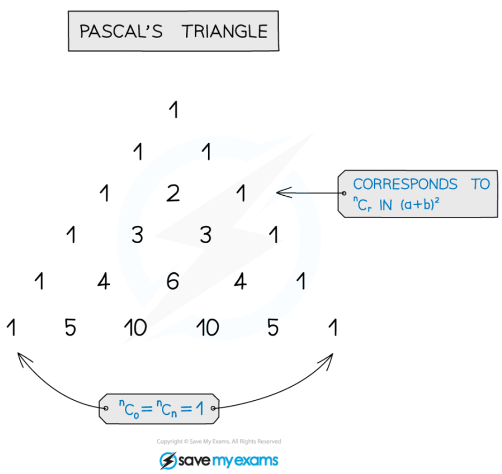
$\boxed{\text{Coefficient of } x^3 = 84}$

Pascal's Triangle

YOUR NOTES
↓

What is Pascal's Triangle?

- Pascal's triangle is a way of arranging the binomial coefficients and neatly shows how they are formed
  - Each term is formed by adding the two terms above it
  - The first row has just the number 1
  - Each row begins and ends with a number 1
  - From the third row the terms in between the 1s are the sum of the two terms above it



How does Pascal's Triangle relate to the binomial theorem?

- Pascal's triangle is an alternative way of finding the binomial coefficients, ^nC_r
  - It can be useful for finding for smaller values of n without a calculator
  - However for larger values of n it is slow and prone to arithmetic errors
- Taking the first row as zero, $(^0C_0 = 1)$, each row corresponds to the n^{th} row and the term within that row corresponds to the r^{th} term

Exam Tip

- In the non-calculator exam Pascal's triangle can be helpful if you need to get the coefficients of an expansion quickly, provided the value of n is not too big

YOUR NOTES

Worked Example

Write out the 7th row of Pascal's triangle and use it to find the value of ${}^{6}C_{4}$.

7th row of Pascal's Triangle:

${}^{0}C_{0}$

1

${}^{1}C_{0}$ $1 \quad 1$ ${}^{1}C_{1}$

$1 \quad 2 \quad 1$

$1 \quad 3 \quad 3 \quad 1$

$1 \quad 4 \quad 6 \quad 4 \quad 1$

$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$

$+ \quad + \quad + \quad + \quad +$

7th row $\rightarrow$ $1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1$

${}^{6}C_{0}$ ${}^{6}C_{4}$

7th row of Pascal's Triangle: 1, 6, 15, 20, 15, 6, 1

$${}^{6}C_{4} = 15$$