5.4 Further Integration

Question Paper

Course	DP IB Maths
Section	5. Calculus
Topic	5.4 Further Integration
Difficulty	Very Hard

Time allowed: 120

Score: /92

Percentage: /100

Question 1

Consider the function f defined by $f(x) = (x^2 - 3x + 2)(x + 2),\ x \in \mathbb{R}$.

Calculate the area of the region enclosed by the graph of $y = f(x)$ and the x-axis.

[8 marks]

Question 2a

(a) Find the indefinite integral for

$$\int \sin\left(\frac{\sqrt{3}}{2}x\right)\,\mathrm{d}x$$

[2 marks]

Question 2b

(b) Find the indefinite integral for

$$\int \frac{7}{e^{4x-9}}\,dx$$

[2 marks]

Question 2c

(c) Find an expression for y given that

$$\frac{dy}{dx} = \cos\left(2\left(\frac{\pi}{8} - x\right)\right)$$

[2 marks]

Question 3a

(a) Find the exact value of

$$\int_{-4}^{-1} -\frac{7}{5x}\,dx$$

[3 marks]

Question 3b

(b) Find the definite integral

$$\int_{-\frac{\pi}{3}}^{0} \sin\left(\frac{\pi}{3} - 2x\right)\,\mathrm{d}x$$

[3 marks]

Question 3c

(c) Find an expression for y given that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x\mathrm{e}^{x^2-2}$$

and also that $y = 3$ when $x = -\sqrt{2}$.

[3 marks]

Question 4

Use a suitable substitution to show that

$$\int_3^4 \frac{x^3}{2(x+2)(x-2)}\,\mathrm{d}x = \frac{7}{4} + \ln\left(\frac{12}{5}\right)$$

[7 marks]

Question 5a

Let I be the definite integral defined by

$$I = \int_{\frac{a}{k}}^{\frac{b}{k}} \sin^2(k\theta)\, \mathrm{d}\theta$$

where a, b and k are real constants such that $a \le b$ and $k > 0$.

(a) Show that

$$I = \frac{1}{2k}\left[(b-a) - \frac{1}{2}(\sin(2b) - \sin(2a))\right]$$

[7 marks]

Question 5b

(b) Hence find the exact values of

(i) $$\int_{\frac{\pi}{12}}^{\frac{\pi}{3}} \sin^2(2\theta)\,\mathrm{d}\theta$$

(ii) $$\int_{-\frac{5\pi}{2}}^{10\pi} \sin^2\left(\frac{\theta}{5}\right)\mathrm{d}\theta$$

[4 marks]

Question 6a

(a) Explain why

$$\frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$

[2 marks]

Question 6b

(b) Use the result from part (a) to show that

$$\int_0^{\sqrt{\frac{\pi}{6}}} \frac{x}{\tan\left(x^2 - \frac{2\pi}{3}\right)}\, dx = -\frac{1}{2}\ln\left(\frac{\sqrt{3}}{2}\right)$$

[7 marks]

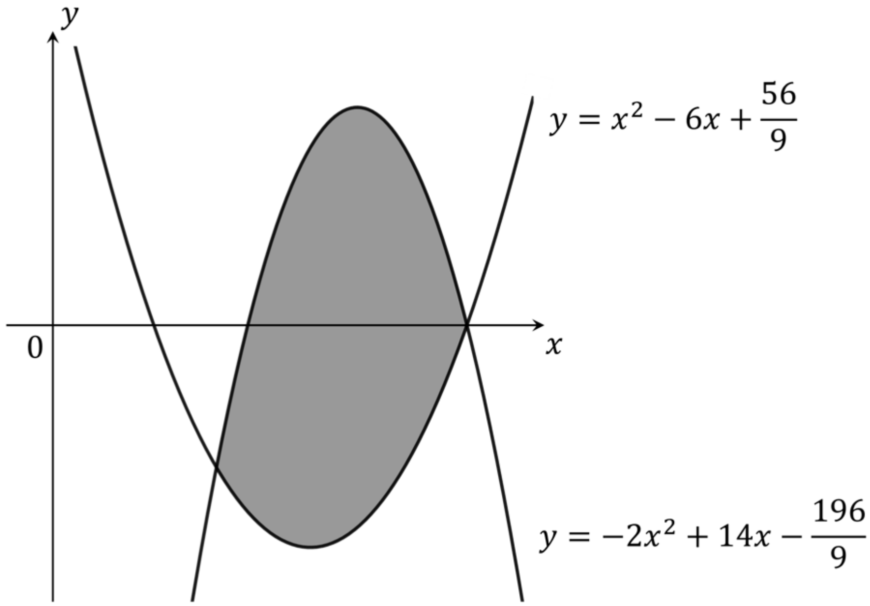
Question 6c

(c) Explain why the value of the integral found in part (b) is a positive number.

[1 mark]

Question 7a

The diagram below shows a sketch of part of the curves with equations $y = x^2 - 6x + \frac{56}{9}$ and $y = -2x^2 + 14x - \frac{196}{9}$.



The shaded region in the diagram is the area bounded by the two curves.

(a) By first showing that the area of the shaded region is given by

$$\int_2^{\frac{14}{3}} (20x - 3x^2 - 28)\,\mathrm{d}x$$

calculate the exact area of the shaded region

[6 marks]

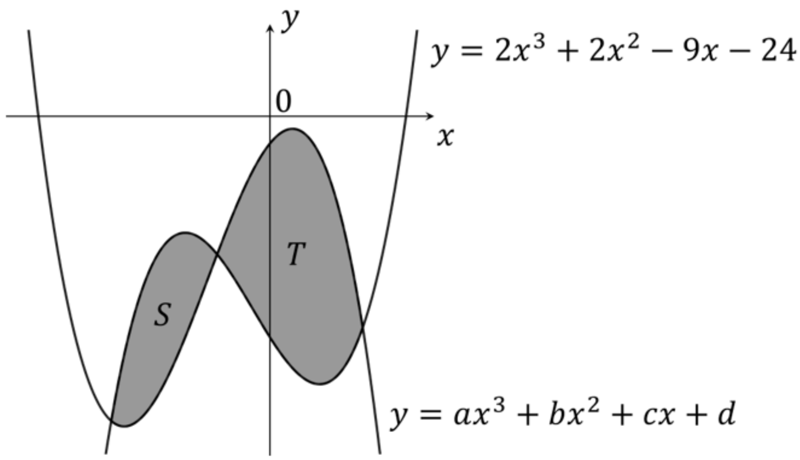
Question 7b

(b) Explain why your answer to part (a) is not affected by the fact that the shaded region is partially above and partially below the x-axis.

[2 marks]

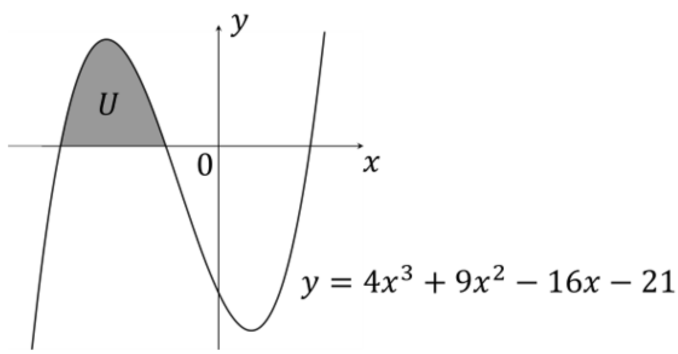
Question 8

The diagram shows a sketch of part of the curves with equations $y = 2x^3 + 2x^2 - 9x - 24$ and $y = ax^3 + bx^2 + cx + d$, where a, b, c and d are constants with $a \neq 0$.



The x-coordinates of the points of intersection of the two curves are p, q and r, where $p < q < r$. Region S is the region enclosed by the two curves between $x = p$ and $x = q$, while region T is the region enclosed by the two curves between $x = q$ and $x = r$.

The diagram below shows a sketch of part of the curve with equation $y = 4x^3 + 9x^2 - 16x - 21$.



The curve intersects the x-axis at the points $(p, 0)$, $(q, 0)$ and $(r, 0)$, and region U is the region enclosed by the curve and the x-axis between $x = p$ and $x = q$.

Given that the areas of regions S and U are equal, calculate the total area enclosed by the two curves in the first diagram. Be sure to provide a suitable justification for your answer.

[7 marks]

Question 9a

Consider the function $h(x)$ such that

$$\int_{6}^{-3} h(x)\,dx = 14 \quad \text{and} \quad \int_{2}^{5} h(x)\,dx = 14$$

(a) Find

(i) $\displaystyle\int_{5}^{2} h(x)\,dx$

(ii) $\displaystyle\int_{-2}^{-2} h(x)\,dx$

(iii) $\displaystyle\int_{-3}^{2} h(x)\,dx + \int_{5}^{6} h(x)\,dx$

[5 marks]

Question 9b

(b) Find

$$\int_{6}^{-3} \frac{7-3h(x)}{4}\,\mathrm{d}x$$

[3 marks]

Question 9c

(c) Given that $h(2)=3$ and $h(5)=4$, find

$$\int_{2}^{5} h(x)(4h'(x)-\pi)\,\mathrm{d}x$$

[4 marks]

Question 10a

(a) Show that $5w^3-21w^2+16=(5w+4)(w^2-5w+4)$.

[2 marks]

Question 10b

A function f is defined by $f(x) = -\frac{16}{x^2} - 5x + 21,\ x \neq 0.$

Let I be the definite integral defined by

$$I = \int_1^a f(x)\,\mathrm{d}x$$

where $a > 1$ is a constant.

(b) Determine the value of I, giving your answer in terms of a.

[4 marks]

Question 10c

(c) Hence, or otherwise, determine the value of a which maximises the value of I, and calculate the value of I when a takes that value.

[8 marks]