4.9 Further Normal Distribution (inc Central Limit Theorem)

Question Paper

Course	DP IB Maths
Section	4. Statistics & Probability
Topic	4.9 Further Normal Distribution (inc Central Limit Theorem)
Difficulty	Hard

Time allowed: 80

Score: /60

Percentage: /100

Question 1a

Caleb owns a fishing lake. He catches 22 fish from the lake. He measures the fish that he catches and finds that they have mean length of 34.2 cm with standard deviation 7 cm. It is assumed that the lengths of the fish are normally distributed.

a)
Find a 95% confidence interval for the population mean.

[3 marks]

Question 1b

b)
Caleb produces an advert for his fishing lake, stating that the average length of fish is 37 cm. Comment on Caleb's claim.

[1 mark]

Question 1c

c)
Explain whether or not you needed to use the central limit theorem in your answer to part (a).

[1 mark]

Question 2

Dan enjoys a cup of coffee in his favourite mug every morning. He wants to check that the amount of coffee dispensed by his coffee machine stays consistent.

He measures the volume of coffee in his mug each morning and records the data in a spreadsheet over 31 days. The mean of Dan's data is 219 ml and the standard deviation is 4.6 ml.

Dan decides that the machine needs its settings adjusted if the amount of coffee it is dispensing on average is different to 220 ml.

Using a 95% confidence interval for the population mean, decide if Dan needs to adjust the settings on the coffee machine, justifying your answer.

[4 marks]

Question 3a

SME Juices manufactures 250 ml cartons of apple juice. The quality control manager needs to make sure that the volume in each carton is suitably close to the advertised volume of 250 ml. She takes a random sample of 34 cartons and measures the volume of juice that they contain. It is assumed that the volume of apple juice in each carton follows a normal distribution. Her findings are summarised below.

$$\Sigma x = 8495 \qquad s_n^{\,2} = 14.684$$

a)
Find unbiased estimates for the mean and variance of the population.

[2 marks]

Question 3b

b)
Find a 95% confidence interval for the population mean.

[2 marks]

Question 3c

c)
A customer complains that the mean volume in the apple juice cartons is 245 ml. State whether the customer's complaint is justified, giving a reason for your answer.

[2 marks]

Question 3d

A manager says that customers are only likely to complain if there is less than 246 ml of juice in a carton. He sets a target of less than 3% of customers making a complaint.

d)
Assuming that the estimated mean and variance of the population are in fact the actual mean and variance
i)
find the probability that a carton contains less than 246 ml of juice
ii)
suggest whether the manager's target is likely to be met.

[2 marks]

Question 4a

At an alligator sanctuary, the lengths of alligators, L metres, are assumed to be normally distributed with mean and standard deviation σ.

A random sample of 12 alligators are safely captured so their health can be monitored. The sample can be summarised as follows

$$\sum l = 39 \quad \sum l^2 = 152$$

a)
Find an unbiased estimate for the mean of L.

[1 mark]

Question 4b

b)

Use the formula $s_{n-1}{}^2 = \dfrac{\Sigma x^2 - \left(\frac{\Sigma x}{n}\right)^2}{n-1}$ to find an unbiased estimate of the variance of L.

[1 mark]

Question 4c

c)
Find a 90% confidence interval for the mean length of alligators.

[2 marks]

Question 4d

d)
Explain how the sanctuary could obtain a smaller 90% confidence interval for the mean length of alligators.

[1 mark]

Question 4e

e)
A vet explains that in a population of alligators, it is likely that there are far more extremely long alligators than extremely short alligators. Explain how this affects the validity of the answer to part (c).

[2 marks]

Question 5a

A veterinary nurse is investigating the weights of cats who attend her clinic. Over 1 week she weighs 34 cats. She records their weights and at the end of the week finds that the mean of her sample is 4.2 kg and the standard deviation is 0.5 kg.

a)
Find a 90% confidence interval for the mean weight of cats visiting her clinic overall.

[3 marks]

Question 5b

She decides to extend her study so that it lasts for a whole month. Her sample now includes 135 cats in total, with a mean of 4.18 kg.

b)
Explain what is likely to happen to the width of the 90% confidence interval as a result of extending her study.

[1 mark]

Question 5c

The veterinary nurse later finds a database containing the whole population of cats who have ever visited the veterinary practice. The database shows that the standard deviation of the weight of the cats is 0.4 kg.

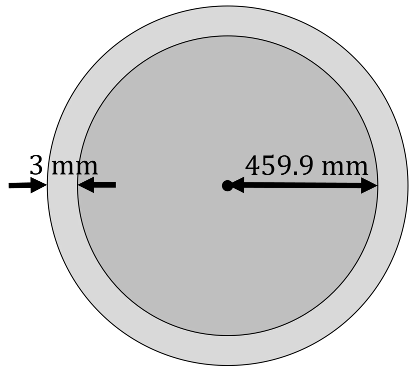
c)
Using this new information, and the sample for the whole month, find a new 90% confidence interval for the mean weight of cats visiting the practice.

[2 marks]

Question 6

A factory manufactures wheels for trains. The radii of the wheels follow a normal distribution. The mean of the radii of the wheels is 459.9 mm and the standard deviation is 0.2 mm.

Once the wheels are made, the circumference of each wheel is coated in a uniform protective layer of thickness 3 mm, as shown in the diagram.



Find, including the protective layer:

i)
the mean of the diameters of the wheels
ii)
the variance of the diameters of the wheels.

[4 marks]

Question 7a

An ecologist in Antarctica is weighing adult from a large colony. The mass of the adult penguins is normally distributed with mean 4.8 kg and standard deviation 1.2 kg.

The ecologist decides that any adult penguins that have a mass less than 3 kg are at risk of malnutrition.

a)
Find the probability that a penguin selected at random from the colony will be considered at risk of malnutrition.

[2 marks]

Question 7b

The ecologist is making a video to show the health of penguins in the colony, and she decides to select 8 of the penguins at random to feature.

b)
Find the probability that the mean mass of the 8 selected penguins would be classed as being at risk of malnutrition.

[3 marks]

Question 7c

c)
Explain why the value for part (b) is lower than part (a).

[1 mark]

Question 8a

The number of goals scored in a soccer match follows a Poisson distribution with an average of 2.5 goals per match.

a)
Find the probability that there are no goals scored in a match.

[1 mark]

Question 8b

Gary is analysing the number of goals scored per match at the World Cup, where there were 64 matches played.

b)
Using the central limit theorem, estimate the probability that the mean number of goals per match at the World Cup is 3 or more.

[3 marks]

Question 8c

Gary wants to analyse the mean number of goals per match, just in the matches involving England. England played in 7 matches at the tournament.

c)
Comment on whether the method used in part (b) would still be suitable.

[1 mark]

Question 9a

In a role-playing game a 12-sided fair dice, numbered 1 to 12, is rolled to determine if a character's magic spell is successful. If the rolled number is 8 or higher; their spell is successful. If the rolled number is less than 8 their spell is unsuccessful. In a particular magical battle, 6 spells are cast.

a)
Define the probability distribution that could be used to model the number of successful spells in the magical battle. State any assumptions that are necessary.

[2 marks]

Question 9b

b)
Find
i)
the mean of this distribution.
ii)
the variance of this distribution.

[2 marks]

Question 9c

c)
During the whole game, there are 48 magical bottles; each involving 6 spells, as modelled previously. Find the probability that there are on average, 2 or fewer successful spells per battle.

[3 marks]

Question 9d

d)
Explain why it is valid to use the central limit theorem for part (c).

[1 mark]

Question 10a

It is suggested that the number of potholes (small holes in the road) in a 1 km stretch of road can be modelled by a Poisson distribution. There are estimated to be 15 potholes per kilometre in the UK on average.

Lucy works in the council and receives a complaint from a local resident, stating that they encountered over 100 potholes on their 5 km journey to work. Lucy decides to model the frequency of potholes using a Poisson distribution.

a)
Using a Poisson distribution, find the probability that there are over 100 potholes on a 5 km stretch of road.

[2 marks]

Question 10b

Lucy claims that there are fewer potholes per 1 km in her local area than the national average. She decides to investigate this by taking a random sample of 50 separate 1 km sections of road.

b)
Lucy calculates the mean number of potholes in a 1 km section of road is 13.5 (10% below the national average).
i)
Find the probability, according to the model, that the mean number of potholes per 1 km stretch of road in her sample is no more than 13.5.
ii)
Suggest, giving a reason, whether Lucy's claim is justified.

[4 marks]

Question 10c

c)
State a possible reason why a Poisson distribution may not be an accurate to model the occurrence of potholes.

[1 mark]