

6.1 Extended Questions

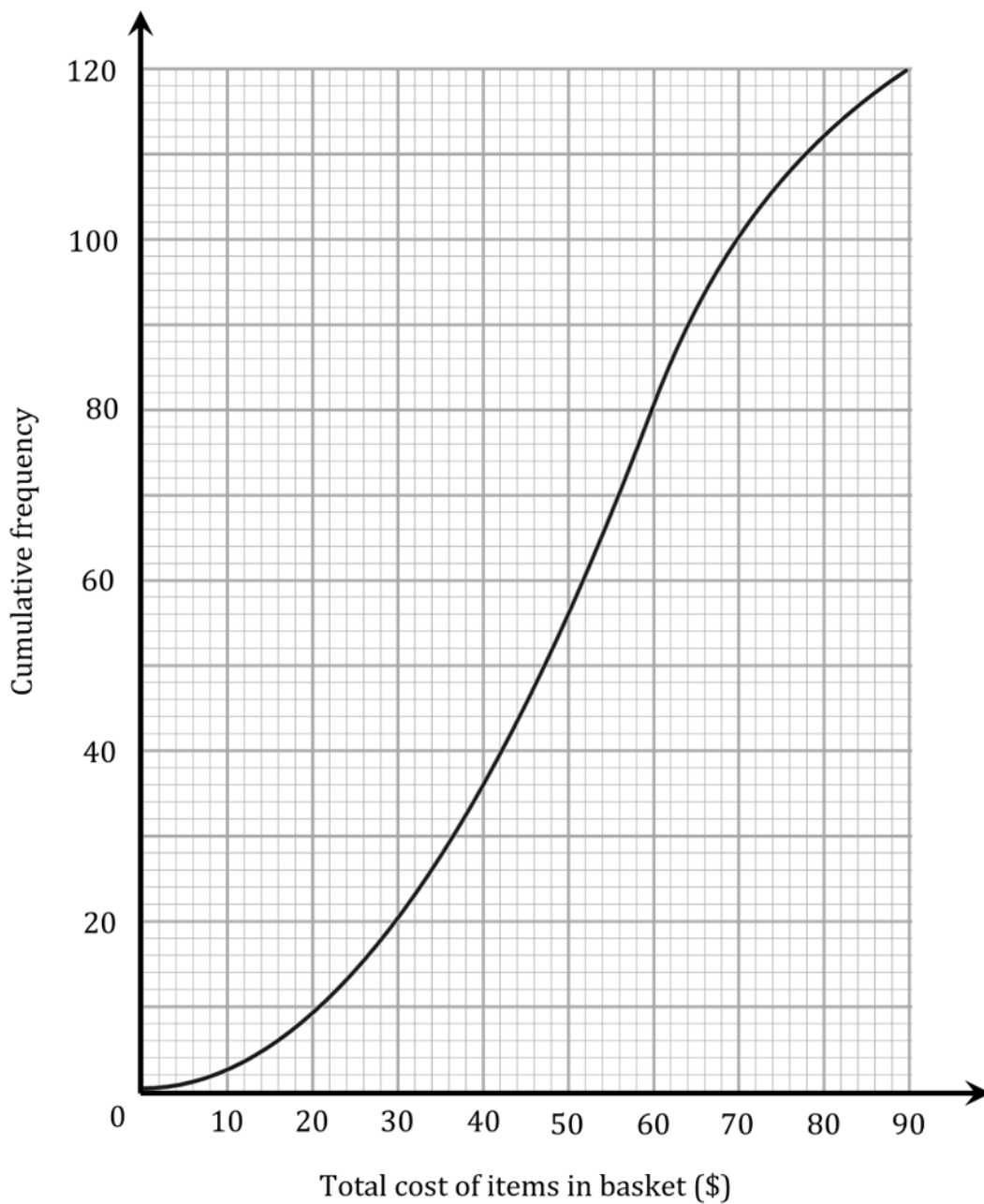
Question Paper

Course	DPIB Maths
Section	6. Extended Questions
Topic	6.1 Extended Questions
Difficulty	Medium

Time allowed: 130
Score: /101
Percentage: /100

Question 1a

A supermarket manager wishes to gather information about the spending habits of the store's customers. During each of his lunchbreaks on Monday through Friday of a given week, he chooses 24 customers at random and notes the total cost, in dollars (\$), of the items in their baskets when they check out at the tills. The results of his survey are represented by the following cumulative frequency graph.



(a) Find the median total cost of the items these customers had in their baskets.

[2 marks]

Question 1b

(b) Find the interquartile range of the total cost of the items these customers had in their baskets.

[3 marks]

Question 1c

(c) Given that two thirds of customers had a total of more than $\$p$ of goods in their baskets, find the value of p .

[3 marks]

Question 1d

The same survey information is represented by the following table:

Total cost (\$$m$) of goods in basket	$0 < m \leq 20$	$20 < m \leq 40$	$40 < m \leq 70$	$70 < m \leq 90$
Frequency	9	q	r	20

(d) Find the value of q and the value of r .

[4 marks]

Question 1e

In an average week, the manager estimates that the store has a total of 3600 customers.

(e) Use the results of the manager's survey to estimate the number of customers in a week who have goods totalling more than \$50 in their baskets.

[3 marks]

Question 1f

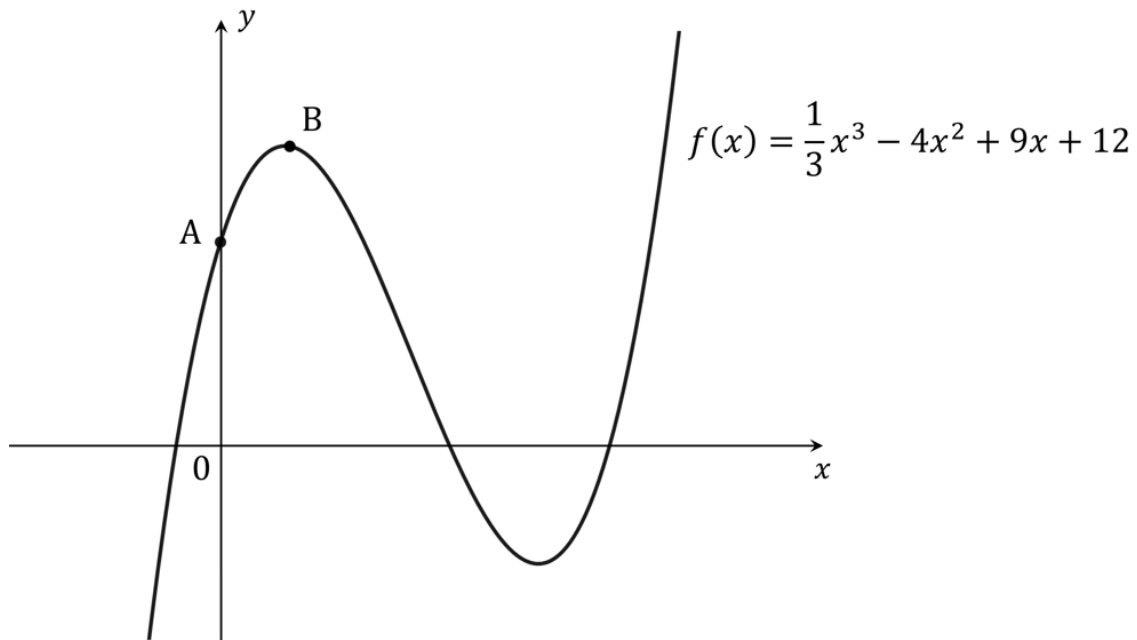
- (f) (i) Explain why the manager's survey sample might not provide an accurate representation of the spending habits of *all* the shop's customers.
- (ii) Suggest a sampling method that might obtain a more representative sample.

[2 marks]

Question 2a

The diagram below shows a part of the graph of the function

$$f(x) = \frac{1}{3}x^3 - 4x^2 + 9x + 12$$



(a) Point A is the point of intersection between the graph and the y -axis. Write down the coordinates of point A.

[1 mark]

Question 2b

(b) Find $f'(x)$.

[2 marks]

Question 2c

(c) Using the graph, explain why the equation $f'(x) = 0$ must have exactly two distinct real solutions.

[3 marks]

Question 2d

Point B is the point on the graph with x -coordinate $\frac{8-\sqrt{26}}{2}$.

(d) Given that $\left(\frac{8-\sqrt{26}}{2}\right)^2 = \frac{45-8\sqrt{26}}{2}$, find the gradient of the tangent line to the graph at point B .

[2 marks]

Question 2e

Points C and D are the points on the graph at which the tangent lines are perpendicular to the tangent line at point B .

(e) By first determining the gradient of the tangents at points C and D , find the x -coordinates of points C and D .

[5 marks]

Question 2f

- (f) Given that point C lies between points A and B on the graph, find the equation of the tangent line to the graph at point C.

[4 marks]

Question 3a

After escaping from a research station, a small population of rabbits has become established on an island in the Southern Ocean. Scientists have begun to study this rabbit population, and have determined that the number of rabbits, P , at a time t months after the beginning of the study can be modelled by the function

$$P(t) = \frac{3000}{1 + 99e^{-kt}}$$

Where k is a positive constant.

(a) Determine the number of rabbits on the island at the beginning of the study.

[2 marks]

Question 3b

- (b) (i) Explain what happens to the values of e^{-kt} as t becomes large.
- (ii) Hence determine the maximum number of rabbits that the model predicts the island can support. Be sure to show clear mathematical reasoning to support your answer.

[4 marks]

Question 3c

(c) Show that

$$P'(t) = \frac{3000 \times 99ke^{-kt}}{(1 + 99e^{-kt})^2}$$

[3 marks]

Question 3d

- (d) (i) Use the result from part (c) to show that $P(t)$ is an increasing function for all values of $t \geq 0$.
- (ii) Explain why this does not contradict the result of (b)(ii).

[4 marks]

Question 3e

The model predicts that the population of rabbits will double in the first two months after the beginning of the study.

- (e) (i) Use this information to show that $k = \frac{1}{2} \ln \left(\frac{99}{49} \right)$.
- (ii) Hence find the exact rate of change of the rabbit population at the beginning of the study, as predicted by the model.

[7 marks]

Question 4a

The Strike A Light! matchstick company produces matchsticks with a length, X mm, that is normally distributed with mean 45 and variance σ^2 .

The probability that X is greater than 45.37 is 0.1714.

(a) Find $P(44.63 < X < 45.37)$.

[2 marks]

Question 4b

(b) (i) Find σ , the standard deviation of X .

(ii) Hence, find the probability that a randomly selected matchstick has a length less than 44.5 mm.

[5 marks]

Question 4c

Andrew has a box of Strike A Light! matches with fifteen matchsticks remaining in it. Those matchsticks may be assumed to be a random sample. Let Y represent the number of matchsticks in Andrew's box with lengths less than 44.5 mm.

(c) Find $E(Y)$.

[3 marks]

Question 4d

(d) Find the probability that exactly one of the matchsticks in Andrew's box has a length less than 44.5 mm.

[2 marks]

Question 4e

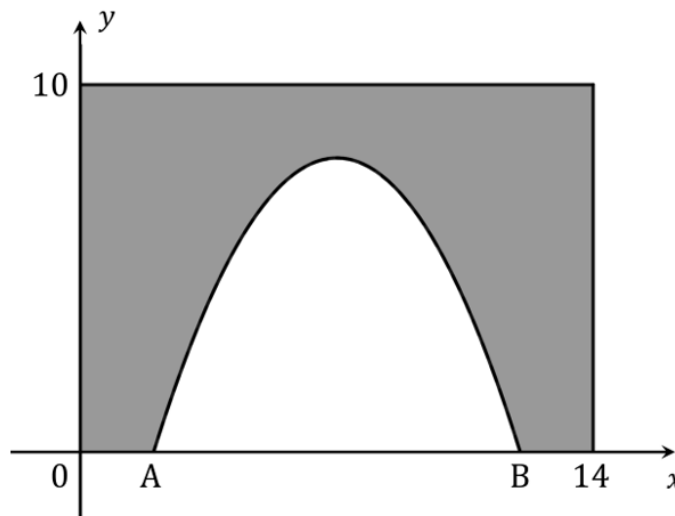
A Strike A Light! matchstick is selected at random and is found to have a length greater than 44.5 mm.

(e) Find the probability that the length of the matchstick is between 44.63 mm and 45.37 mm.

[3 marks]

Question 5a

K. C. Jones & Company produces tunnels for model railroad layouts. Each tunnel has the form of a right prism, and the cross-section of one of the tunnels the company produces is shown in the diagram below. The upper and right-hand borders of the shaded area are parallel to the x -axis and y -axis respectively, and all units are in centimetres.



The shape of the opening of the tunnel may be modelled by the function

$$f(x) = -k(x^2 - 14x + 24)$$

where k is a positive constant.

Points A and B are the points where the tunnel opening meets the x -axis in the diagram.

(a) Find the coordinates of points A and B.

[3 marks]

Question 5b

The maximum height of the tunnel opening above the x -axis is 8 cm.

(b) Use this information to determine the value of k .

[3 marks]

Question 5c

(c) By setting up and solving an appropriate definite integral, show that the area of the tunnel opening is $\frac{160}{3}$ cm². You must use calculus and show the steps of your working.

[4 marks]

Question 5d

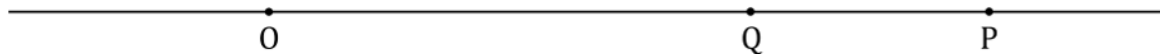
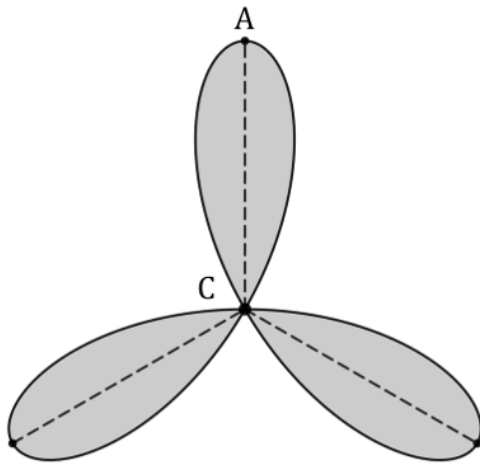
The material from which the tunnel is made has a density of 1060 kg/m^3 .

(d) Given that the mass of the tunnel is 2067 g, find the length of the tunnel.

[5 marks]

Question 6a

Badon Iron Works is building a new ship called the Gargantuan, which will be a full-sized replica of the original RMS Titanic. Eleanor is an engineer at the company, and is involved with construction and testing of the ship's screws (commonly known as 'propellers'). The diagram below depicts one of the ship's screws mounted in the testing facility.



Point C is the centre of the screw, which is fixed in place so that the screw is able to rotate about it. Point A is the marked tip of one of the three identical blades of the screw. Point O is the point on the horizontal floor of the testing facility that lies directly below point C. Points O, A, C, P and Q lie at all times in the same plane.

The height, h m, of point A above the testing facility floor once the screw begins to rotate may be modelled by the function

$$h(t) = 5.59 + 3.6 \cos(k\pi t)$$

where t is the time in seconds since the screw began rotating, and k is a constant.

(a) Use the above information to determine:

- (i) The distance of point A from point C.
- (ii) The height of point C above point O.

[2 marks]

Question 6b

- (b) Given that the tips of the three blades of the screw are located at equal distances from each other around the circumference of a circle with centre C, determine the exact distance of point A from the tip of one of the other blades of the screw.

[3 marks]

Question 6c

When it is rotating, the screw makes 75 complete revolutions every minute.

- (c) Given that the argument of the cosine in the equation for $h(t)$ is measured in radians, use this information to determine the value of the constant k .

[3 marks]

Question 6d

Paul, a mathematician, has been hired as a consultant on the Gargantuan project. Because of his height of 1.96 m, Eleanor is concerned about whether he will be able to walk safely beneath the screw while it is rotating.

(d) Determine whether Eleanor is right to be concerned, giving a mathematical reason for your answer.

[2 marks]

Question 6e

The screw has been locked in place so that point A is at its highest possible position above the floor. Paul is standing at point P, which is at a distance of 9.69 m from point O. He walks towards point O until he arrives at point Q, which is located such that

$$\tan \widehat{OQC} = \frac{3}{2} \tan \widehat{OPC}$$

(e) Determine the distance of point Q from point P.

[3 marks]

Question 6f

- (f) Given that point A remains fixed at its highest possible position above the floor, determine the area of triangle APQ.

[4 marks]