2.3 Functions Toolkit

Question Paper

Course	DP IB Maths
Section	2. Functions
Topic	2.3 Functions Toolkit
Difficulty	Very Hard

Time allowed: 130

Score: /105

Percentage: /100

Question 1a

The functions f and g are defined such that $f(x) = 9x - 3x^2 - 3$ and $g(x) = -\frac{66+2x}{3}$, both for $x \in \mathbb{R}$.

(a) Find $(g \circ f)(x)$, giving your answer in the form $(g \circ f)(x) = a(x-p)(x-q)$.

[3 marks]

Question 1b

(b) Hence, or otherwise, find the x-intercepts of the graph of $y = (g \circ f)(x)$.

[1 mark]

Question 1c

Let $h(x) = 1 - 2x$.

(c) Find the distance between the y-intercept of the graph of $y = (f \circ h)(x)$ and the positive x-intercept of the graph of $y = (g \circ f)(x)$. Your answer should be given as an exact value.

[6 marks]

Question 2a

Let the function f be such that $f(x) = \sqrt{5x^2 - 11x + 6.05}$.

Given that the inverse function f^{-1} exists, and that the domain of f is as large as possible,

(a) suggest a domain for f and write down the corresponding range.

[4 marks]

Question 2b

(b) Based on your answer to part (a), find $f^{-1}(\sqrt{22.05})$.

[2 marks]

Question 3a

Let $f(x) = \sqrt{-3x^2 + 8x + 16}$.

(a) Write down the coordinates of the y-intercept of the graph of $y = f(x)$.

[2 marks]

Question 3b

Given that f has the largest possible valid domain,

(b) find the domain and range of f.

[6 marks]

Question 4a

Let the function f be defined by $f(x) = (2x^2 - 5x - 12)^{-\frac{1}{2}} - k$, where k is a constant and where f has the largest possible valid domain.

(a) Find the domain of f.

[2 marks]

Question 4b

(b) Given that that $\lim_{x \to \infty} f(x) = -7$, find the value of k.

[1 mark]

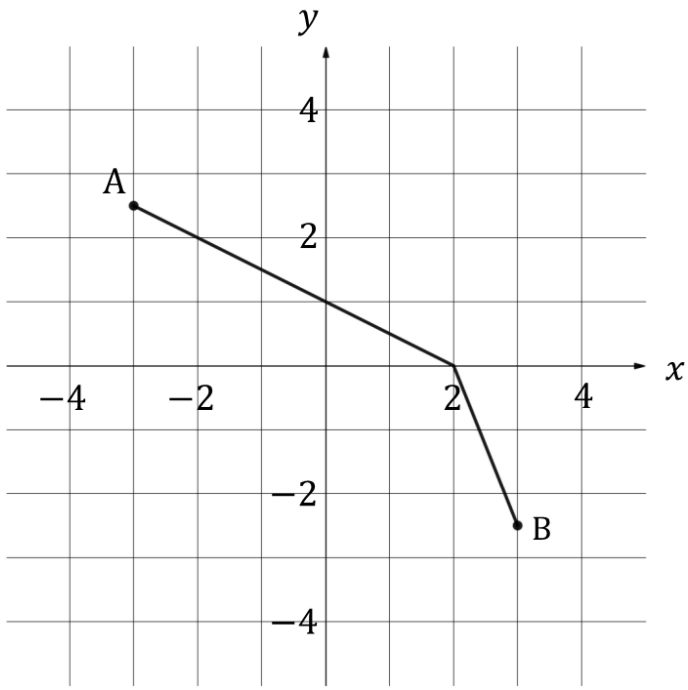
Question 4c

(c) Write down the equations of any vertical and/or horizontal asymptotes on the graph of $y = f(x)$.

[3 marks]

Question 5a

The following diagram shows the graph of $y = f(x)$, for a function f that has the domain $-3 \leq x \leq 3$. Point A has coordinates $(-3, 2.5)$ and point B has coordinates $(3, -2.5)$. The x-intercept of the function is $(2, 0)$ as shown.



f can be written as a piecewise function, where each of the two pieces is a linear function and where the domain of the first function is $-3 \leq x \leq 2$.

(a) Write down $f(x)$ as a piecewise function.

[4 marks]

Question 5b

(b) Sketch the graph of $y = f^{-1}(x)$ on the same grid above.

[3 marks]

Question 6a

Consider the function h defined by $h(x) = -4x^2 + 24x + 8,\ x \in \mathbb{R}$.

(a) Rewrite $h(x)$ in the form $a(x+b)^2 + c$, where $a, b, c \in \mathbb{Z}$.

[2 marks]

Question 6b

(b) Given that $f(x) = (x-3)^2$ and that $(g \circ f)(x) = h(x)$, find $g(x)$.

[3 marks]

Question 7a

The functions f and g are defined such that $f(x) = \frac{3-2x}{5}$ and $g(x) = 4x - 7$, both for $x \in \mathbb{R}$.

(a) Giving your answers in the form $y = mx + c$, find

(i) $(g \circ f)(x)$

(ii) $(f \circ g)(x)$.

[4 marks]

Question 7b

(b) Describe a single transformation that would map the graph of $y = (g \circ f)(x)$ onto the graph of $y = (f \circ g)(x)$.

[2 marks]

Question 7c

(c) Given that $(g \circ f)^{-1}(p) = 2$, find the value of p.

[3 marks]

Question 8a

Let the functions f and g be defined by $f(x) = \frac{9}{4}x^2 - 1$ and $g(x) = x^2 - 2$, both for $x \geq 0$.

(a) Find

(i) $f^{-1}(x)$

(ii) $g^{-1}(x)$.

[2 marks]

Question 8b

(b) Find $(f \circ g)(x)$ in the form $ax^4 + bx^2 + c$.

[2 marks]

Question 8c

(c) Solve the equation $(f \circ g)(x) = 0$.

[3 marks]

Question 9a

A rectangle has length $l = 4x$ and width $w = x$.

(a) Find an expression for

(i) the perimeter of the rectangle, P, in terms of x.

(ii) the area of the rectangle, A, in terms of x.

[2 marks]

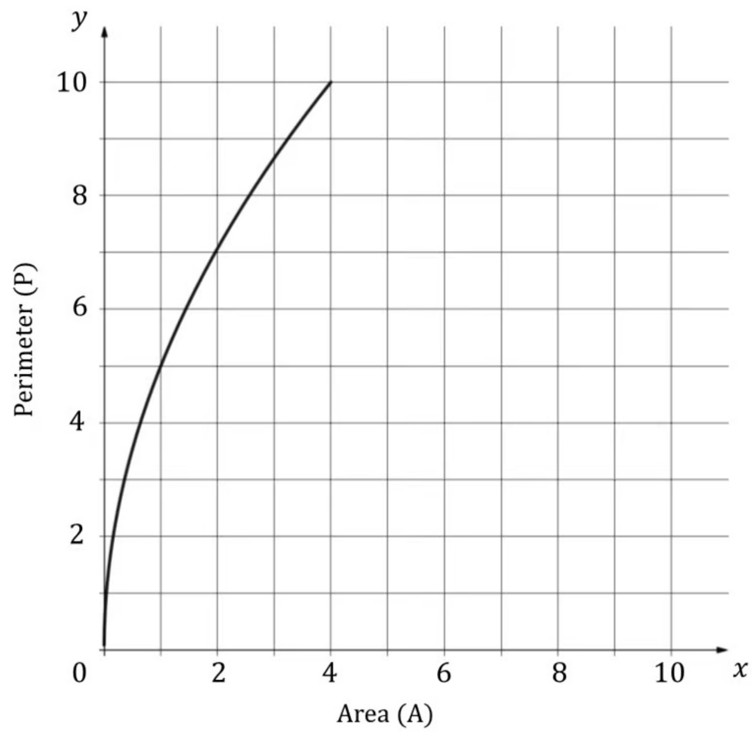
Question 9b

(b) Show that $\mathrm{P(A)} = 5\sqrt{\mathrm{A}}$.

[2 marks]

Question 9c

The graph of the function P, for $0 \le A \le 4$, is shown below.



(c) On the grid above, draw the graph of the inverse function P^{-1}.

[3 marks]

Question 10a

Consider the function f defined by $f(x) = x^2 - 6x + 10,\ x \leq p,$ where p is the largest value such that f has an inverse.

(a) (i) Find the value of p.

(ii) On the same set of axes, sketch the graphs of f and f^{-1}.

(iii) Write down the domain and range of f^{-1}.

[5 marks]

Question 10b

(b) Find the inverse function f^{-1}.

[3 marks]

Question 10c

Let the function g be defined by $g(x) = x^2 - 6x + 10,\ x \in \mathbb{R}$.

(c) (i) Solve $(g \circ f)(x) = 2$.

(ii) Solve $(f \circ g)(x) = 2$.

[4 marks]

Question 11a

Consider the function $f(x) = ax^4 + bx^3 + cx^2 + dx + e$ where $a, b, c, d, e \in \mathbb{R}$.

(a) Show that:

(i) if f is even then $b = d = 0$.

(ii) if f is odd then $a = c = e = 0$.

[3 marks]

Question 11b

Consider the function g defined by $g(x) = (3x + p)(x - 2)(qx + 1)(2x + 3)$ where p and q are real constants.

(b) Find the possible values of p and q in the case where g is an even function.

[5 marks]

Question 11c

(c) Use proof by contradiction to show that g can never be an odd function.

[4 marks]

Question 12a

Consider the function f defined by $f(x) = \frac{2x-5}{3x+k}$, $x \in \mathbb{R}$, $x \neq -\frac{k}{3}$.

(a) In the case where f is self-inverse, find the value of k.

[4 marks]

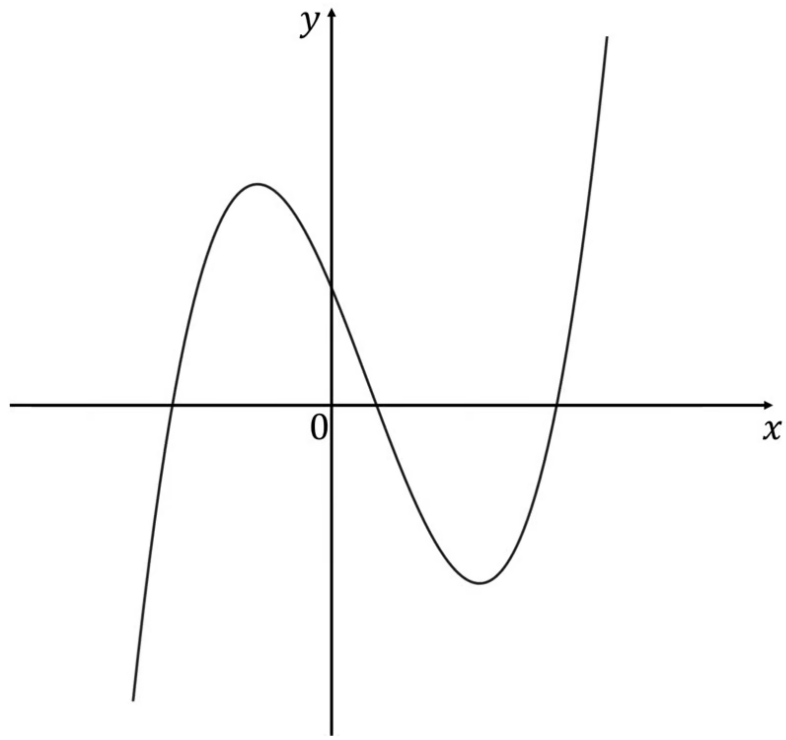
Question 12b

(b) In the case when the graphs of f and f^{-1} intersect at exactly one point, find the possible values of k.

[5 marks]

Question 13a

A part of the graph of the function $f(x) = 2x^3 - 3x^2 - 12x + 8,\ x \in \mathbb{R}$ is shown below.



(a) Explain why f does not have an inverse.

[1 mark]

Question 13b

The domain of f is now restricted to $a \leq x \leq b$ where $a < 0$ and $b > 0$. a and b are chosen so that f has an inverse and the interval $[a, b]$ is as large as possible.

(b) Find the domain and range of f^{-1}.

[6 marks]