5.11 MacLaurin Series

Question Paper

Course	DP IB Maths
Section	5. Calculus
Topic	5.11 MacLaurin Series
Difficulty	Very Hard

Time allowed: 130

Score: /102

Percentage: /100

Question 1a

(a)
Find the first three non-zero terms of the Maclaurin series for $\tan x$ in ascending powers of x.

[6 marks]

Question 1b

(b)
Confirm that the result from part (a) gives the same type of function – either even or odd – as $\tan x$.

[2 marks]

Question 1c

(c)
Hence approximate the value of $\tan 1$

(i)
by substituting the value $x = 1$

(ii)
by substituting another positive value of x .

[4 marks]

Question 1d

(d)

(i)
Compare the approximations found in part (c) to the exact value of $\tan 1$.

(ii)
Explain briefly the reason for the difference in accuracy between the two approximations.

[4 marks]

Question 2a

(a)

Find the first four non-zero terms of the Maclaurin series for e^{-2x} in ascending powers of x.

[4 marks]

Question 2b

(b)

Hence approximate the value of $\sqrt{e}$ and compare this approximation to the exact value.

[3 marks]

Question 2c

(c)

Explain how the accuracy of the Maclaurin series approximation in part (b) could be improved.

[1 mark]

Question 3a

(a)

Find the Maclaurin series for $e^x\left(\sin 3x + \cos\sqrt{x}\right)$ in ascending powers of x, up to and including the term in x^3.

[5 marks]

Question 3b

(b)
Hence find the first three non-zero terms, in ascending powers of x, of the Maclaurin series for

$$e^x\left(2\sin 3x + 6\cos 3x + 2\cos\sqrt{x} - \frac{\sin\sqrt{x}}{\sqrt{x}}\right)$$

[4 marks]

Question 4a

Consider the function f defined by $f(x) = e^{3x}\cos 2x$.

(a)
Show that $f''(x) = pf(x) + pf'(x)$, where p and q are constants to be determined.

[5 marks]

Question 4b

(b)
Hence find the Maclaurin series for $f(x)$ in ascending powers of x, up to and including the term in x^5.

[3 marks]

Question 4c

(c)
Show that $\int f(x)\mathrm{d}x = \frac{e^{3x}}{13}(2\sin 2x + 3\cos 2x) + c$.

[7 marks]

Question 4d

(d)

Hence find the first seven terms, in ascending powers of x, of the Maclaurin series for $e^{3x}(2\sin 2x + 3\cos 2x)$.

[4 marks]

Question 5a

(a)

Find the Maclaurin series for $e^{\frac{1}{2}x^2}$ in ascending powers of x, up to and including the term in x^8.

[4 marks]

Question 5b

The probability density function for the random variable $X \sim \mathrm{N}(0,1)$ is

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

(b)
Use the result of part (a) to find an approximation for the probability $\mathrm{P}(0 \leq X \leq 1)$.

[3 marks]

Question 5c

(c)
Determine the percentage error of your approximation from part (b).

[3 marks]

Question 6

Consider the function f defined by

$$f(x) = \frac{1}{\sqrt{1-2x^2}}$$

By first determining the Maclaurin series of $f(x)$ in ascending powers of x, up to and including the term in x^6, show that

$$\sin\frac{\pi}{4} \approx 0.70710675$$

Be sure to justify that the Maclaurin series is valid for the value of x used to produce your approximation.

[9 marks]

Question 7a

Consider the differential equation

$$y' = \cos x + xy^2$$

together with the initial condition $y(0) = 1$.

(a)
Find expressions for y'', y''', $y^{(4)}$ and $y^{(5)}$. Each should be given in terms of x and y and of lower-order derivatives of y.

[5 marks]

Question 7b

Let $f(x)$ be the solution to the differential equation above with the given boundary condition, so that $y = f(x)$.

(b)
Find the first six terms in ascending powers of x of the Maclaurin series for $f(x)$.

[7 marks]

Question 7c

(c)
Hence find an approximation for the value of y when $x = 0.1$.

[2 marks]

Question 8a

Consider the differential equation

$$y' = \frac{y}{x+1} + 1, \qquad x > -1$$

with the initial condition $y(0) = -1$.

(a)
By first finding expressions for y'', y''', and $y^{(4)}$ in terms of x, y and lower-order derivatives of y, find a Maclaurin series for the solution to the differential equation with the given boundary condition, in ascending powers of x up to and including the term in x^4.

[9 marks]

Question 8b

(b)

Solve the differential equation with the given boundary condition analytically to find an exact solution in the form $y = f(x)$.

[5 marks]

Question 8c

(c)

Find the first four non-zero terms of the Maclaurin series for the answer to part (b), and confirm that they match those in the answer to part (a).

[3 marks]